Control Barrier Functions: Theory Overview and Design Using Dynamic Safety Margins

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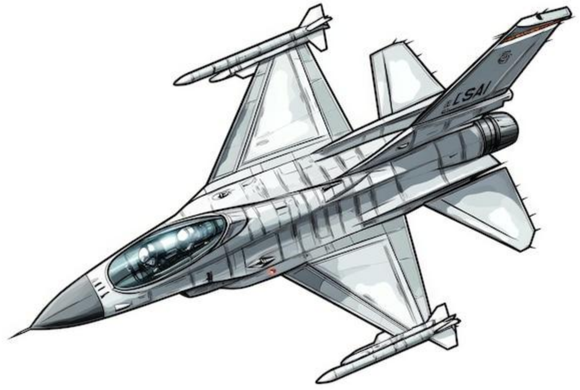
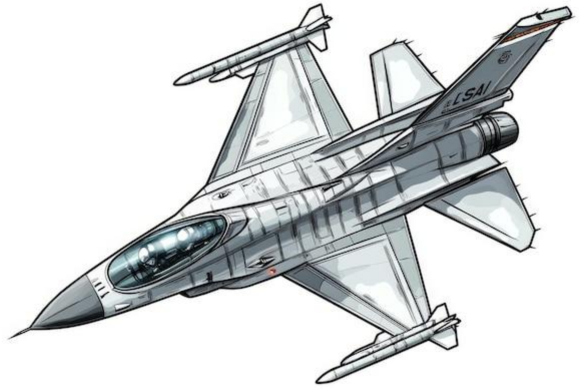
Introduction



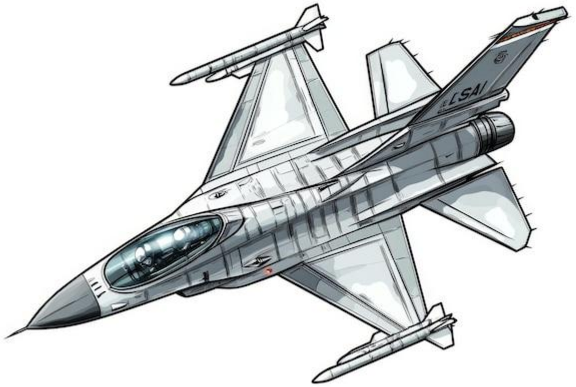
Introduction



Stability

Essentially solved

Introduction



Stability

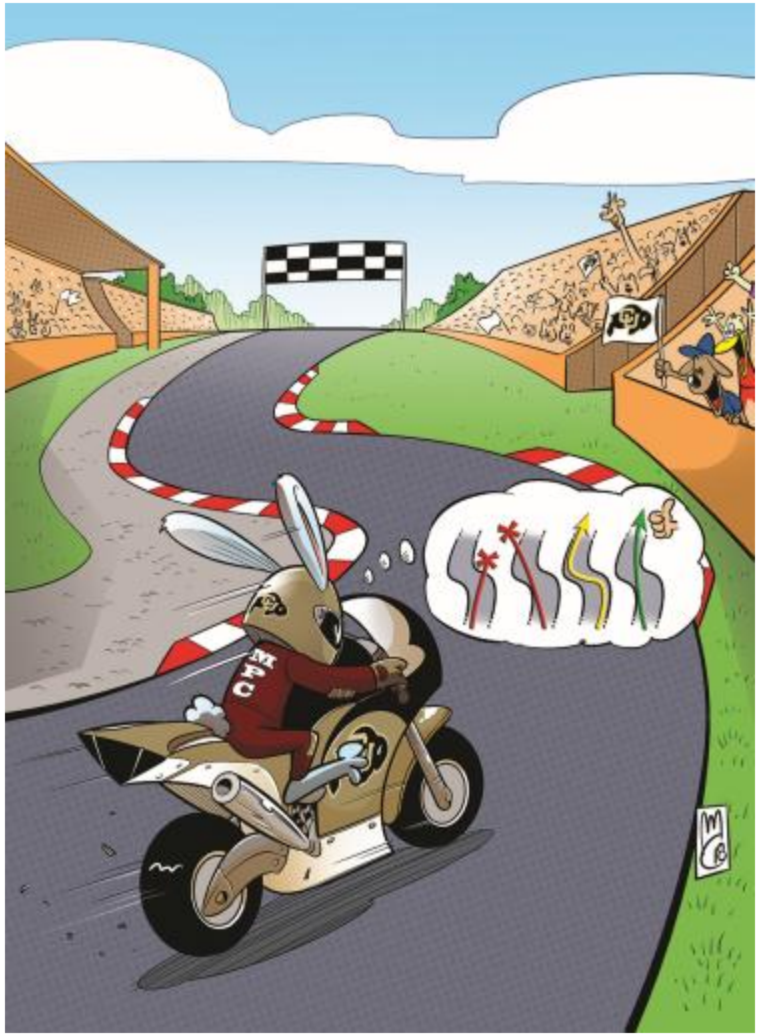
Essentially solved

Safety

Under study
(Workshop)

Constrained Control

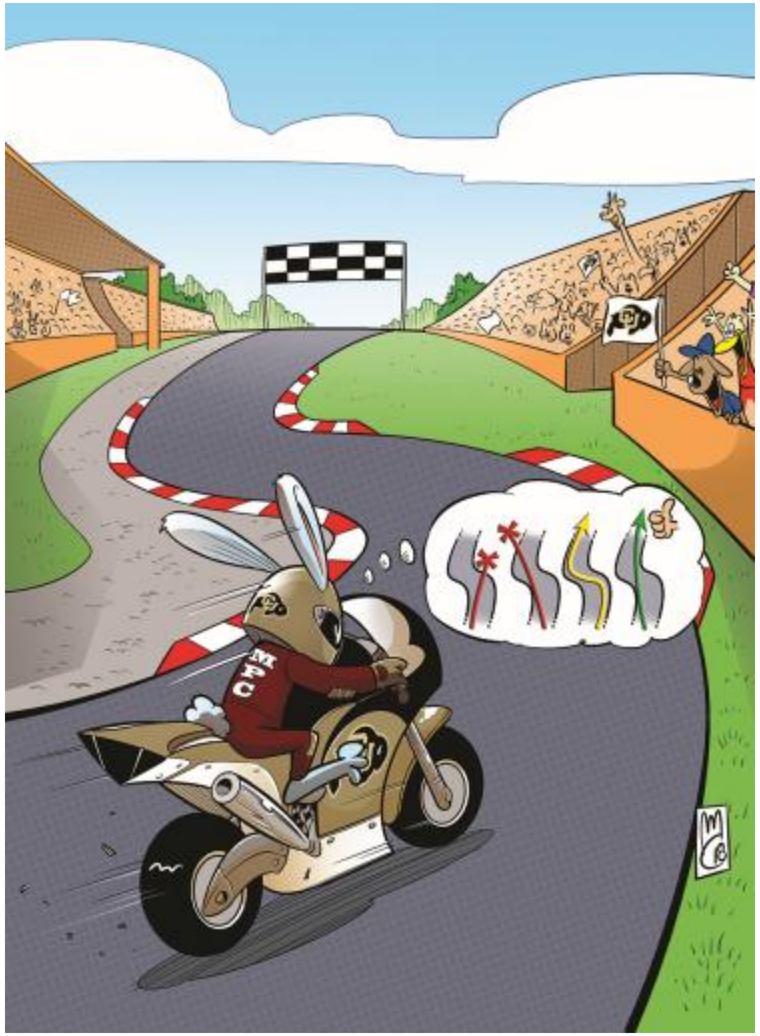
- **Model Predictive Control**
  Full control law redesign
  High computational cost

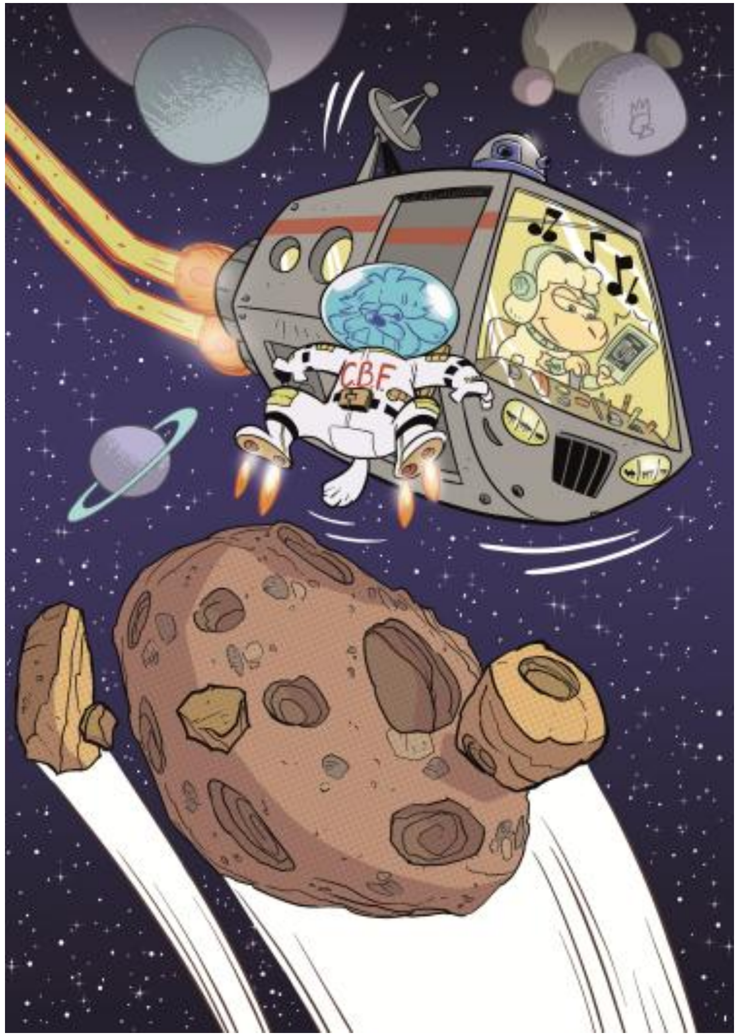


Constrained Control

- **Model Predictive Control**
  Full control law redesign
  High computational cost

- **Safety Filters**
  Keep legacy controller
  Lower computational cost





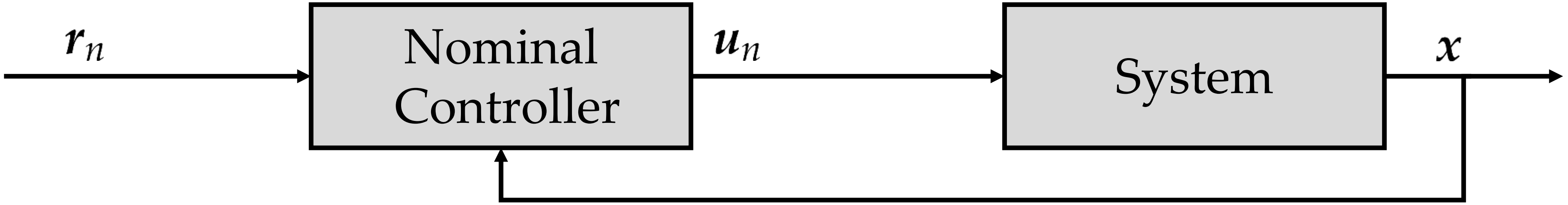
Safety Filters

Family of constrained controllers that use a two-step approach:

Safety Filters

Family of constrained controllers that use a two-step approach:

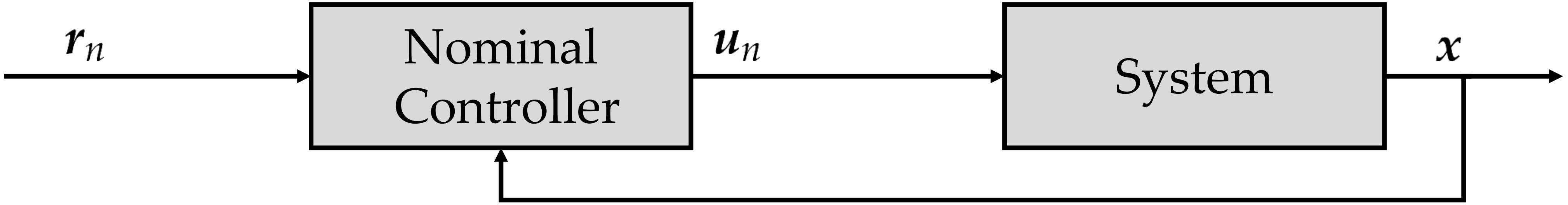
1. Design a control law for the **Unconstrained** system,



Safety Filters

Family of constrained controllers that use a two-step approach:

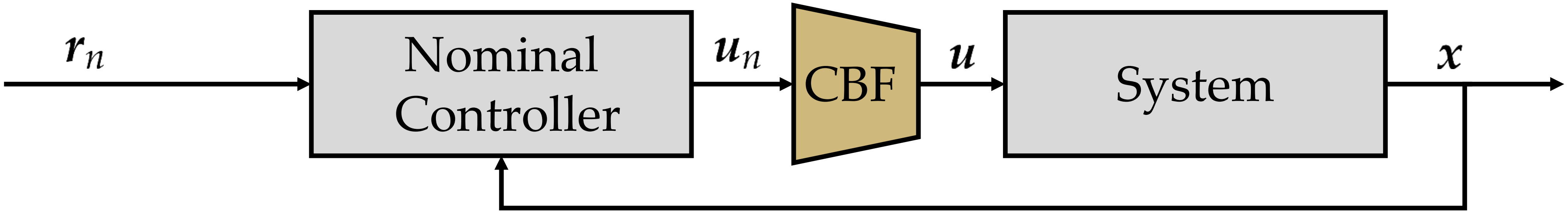
1. Design a control law for the **Unconstrained** system,
2. Introduce an **Add-on** unit for constraint enforcement.



Safety Filters

Family of constrained controllers that use a two-step approach:

1. Design a control law for the **Unconstrained** system,
2. Introduce an **Add-on** unit for constraint enforcement.
   - **Control Barrier Function:** Filter nominal controller



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Problem: *Constrained Control*

Safety and Invariance

Problem: *Constrained Control*

Given a control-affine system

$$\dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)u(t)$$

Safety and Invariance

Problem: *Constrained Control*

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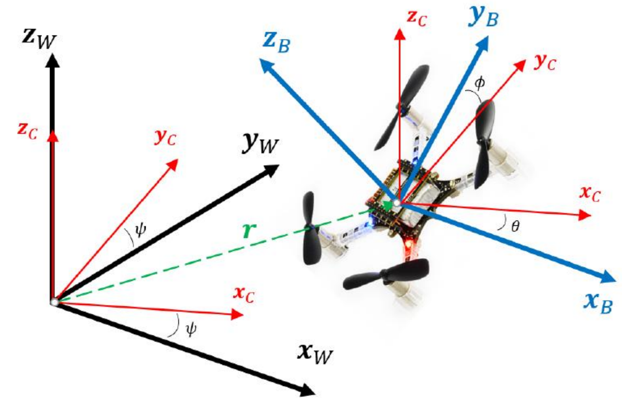
$$\dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)u(t)$$

Quadcopter Dynamics



$$x = \begin{bmatrix} \boldsymbol{r} \\ \xi \\ \dot{\boldsymbol{r}} \end{bmatrix} \qquad \xi = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$$

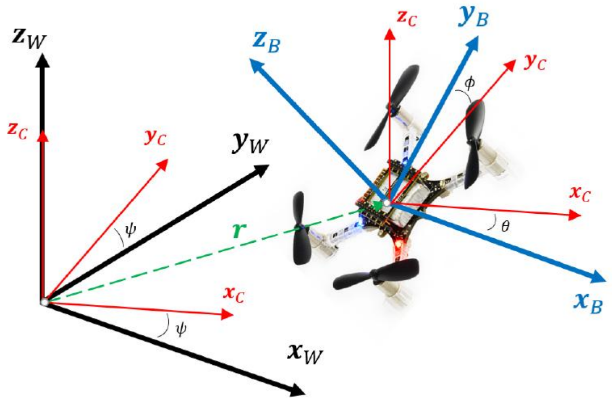
Safety and Invariance

Problem: *Constrained Control*

Given a control-affine system

$$\dot{x}(t) = f\left(x(t)\right) + g\left(x(t)\right)u(t)$$

Quadcopter Dynamics



$$G_\xi(\xi) = \begin{bmatrix} 1 & s\phi t\theta & c\phi t\phi \\ 0 & c\phi & -s\phi \\ 0 & \frac{s\phi}{c\theta} & \frac{c\phi}{c\theta} \end{bmatrix}$$

$$G_T(\xi) = \begin{bmatrix} s\phi s\psi + c\phi s\theta c\psi \\ c\phi s\theta s\psi - s\phi c\psi \\ c\phi c\theta \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{\xi} \\ \dot{\boldsymbol{r}} \end{bmatrix} \qquad \boldsymbol{\xi} = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$$

$$f(\boldsymbol{x}) = \begin{bmatrix} \dot{\boldsymbol{r}} \\ 0 \\ -g_a z_W \end{bmatrix} \qquad g(\boldsymbol{x}) = \begin{bmatrix} 0 & 0 \\ 0 & G_\xi(\boldsymbol{\xi}) \\ G_T(\boldsymbol{\xi}) & 0 \end{bmatrix}$$

Safety and Invariance

Problem: *Constrained Control*

Given a control-affine system
$$\dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)u(t)$$

F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix}\theta & \dot{\theta} & \alpha_w\end{bmatrix}^\top$$
$$\boldsymbol{u} = \begin{bmatrix}\delta_e & \delta_f\end{bmatrix}^\top$$

θ pitch $\quad$ α_w angle of attack $\quad$ δ_e, δ_f control surfaces

Problem: *Constrained Control*

Given a control-affine system

$$\dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)u(t)$$

F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix}\theta & \dot{\theta} & \alpha_w\end{bmatrix}^\top$$

$$\boldsymbol{u} = \begin{bmatrix}\delta_e & \delta_f\end{bmatrix}^\top$$

$$f(\boldsymbol{x}) = A\boldsymbol{x} \quad g(\boldsymbol{x}) = B$$

$$A = \begin{bmatrix}0 & 1 & 0\\ 0 & -0.87 & 43.22\\ 0 & 0.99 & -1.34\end{bmatrix} \quad B = \begin{bmatrix}0 & 0\\ -17.25 & -1.58\\ -0.17 & -0.25\end{bmatrix}$$

Short period approximation

Safety and Invariance

Problem: *Constrained Control*

Given a control-affine system
$$\dot{x}(t) = f\left(x(t)\right) + g\left(x(t)\right)u(t)$$
and constraints
$$\mathcal{X} \subset \mathbb{R}^n,\ \mathcal{U} \subset \mathbb{R}^m,$$

F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix}\theta & \dot{\theta} & \alpha_w\end{bmatrix}^\top$$
$$\boldsymbol{u} = \begin{bmatrix}\delta_e & \delta_f\end{bmatrix}^\top$$
$$f(\boldsymbol{x}) = A\boldsymbol{x} \quad g(\boldsymbol{x}) = B$$

Pitch pointing constraint
$$\mathcal{X} = \{\boldsymbol{x} \in \mathbb{R}^3 \mid |\alpha_w| \leq 4 \text{ deg}\}$$

Control surface deflection limits
$$\mathcal{U} = \{\boldsymbol{u} \in \mathbb{R}^2 \mid |\delta_e| \leq 25 \text{ deg}, |\delta_f| \leq 20 \text{ deg}\}$$

Safety and Invariance

Problem: *Constrained Control*

Given a control-affine system

$$\dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)u(t)$$

and constraints

$$\mathcal{X} \subset \mathbb{R}^n, \ \mathcal{U} \subset \mathbb{R}^m,$$

find a signal $u : [0, \infty) \to \mathcal{U}$ such that $x(t) \in \mathcal{X}$ for all $t \geq 0$.

Safety and Invariance

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find a signal $u : [0, \infty) \to \mathcal{U}$ such that $x(t) \in \mathcal{X}$ for all $t \geq 0$.

Bonus: Trajectory $x(t)$ achieves goal.

(Performance)

Safety and Invariance

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Is there a solution?

Safety and Invariance

Problem: *Constrained Control*

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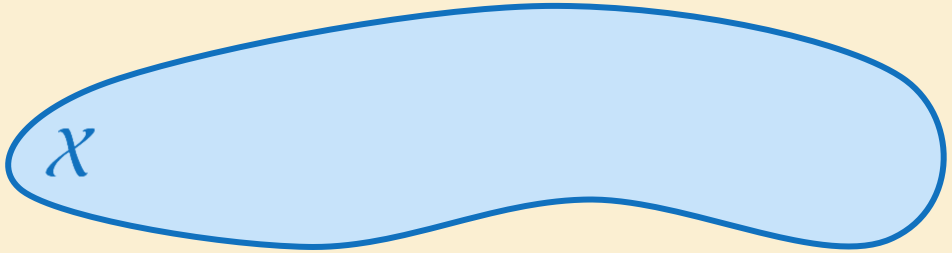
Definition: *Control Invariance*



$\mathcal{X}$ is **control invariant** if for any initial condition $x_0 \in \mathcal{X}$, there exists a signal $u : [0, \infty) \to \mathcal{U}$ such that $x(t) \in \mathcal{X}$.

Solution exists if $\mathcal{X}$ is **control invariant**.

Safety and Invariance

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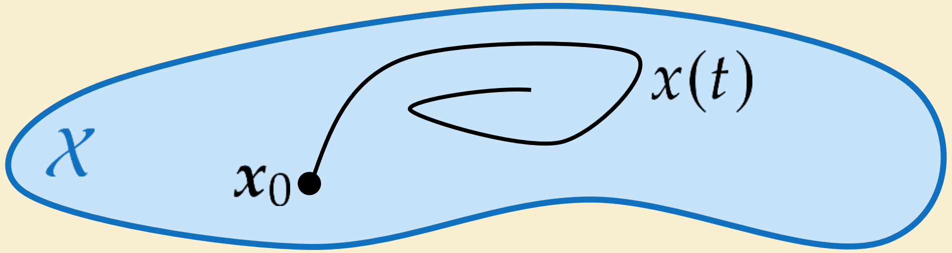

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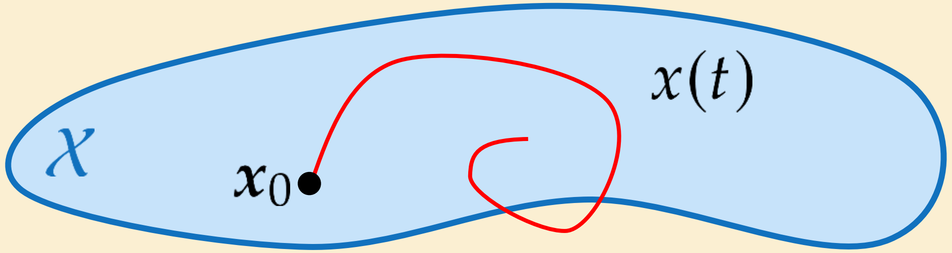

$\mathcal{X}$ is **control invariant** if for any initial condition $x_0 \in \mathcal{X}$, there exists a signal $u : [0, \infty) \to \mathcal{U}$ such that $x(t) \in \mathcal{X}$.

What if $\mathcal{X}$ is not ***control invariant?***

Blanchini, *Automatica* 1999

Safety and Invariance

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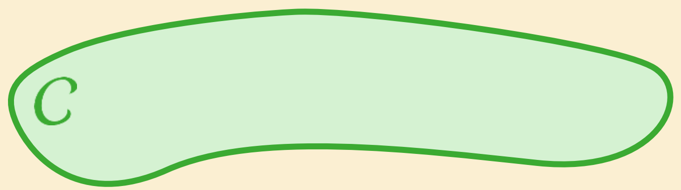

C is **control invariant** if for any initial condition $x_0 \in C$, there exists a signal $u : [0, \infty) \to \mathcal{U}$ such that $x(t) \in C$.

What if $\mathcal{X}$ is not ***control invariant?***

Safety and Invariance

Problem: *Constrained Control*

Given a control-affine system

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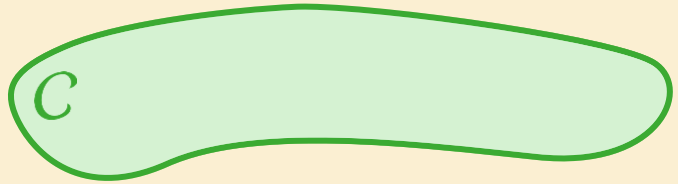
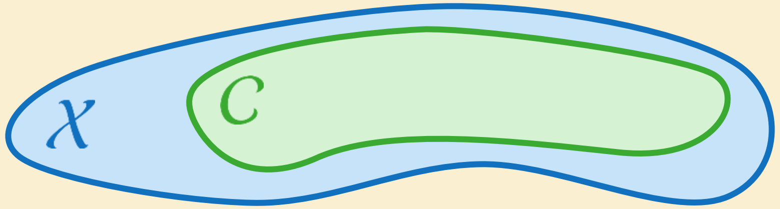


C is **control invariant** if for any initial condition $x_0 \in C$, there exists a signal $u : [0, \infty) \to \mathcal{U}$ such that $x(t) \in C$.

Solution exists if there exists C **control invariant** and **admissible** (i.e., **safe**).

A set C is **admissible** if $C \subset \mathcal{X}$.



Safety and Invariance

Theorem: *Nagumo (simplified)*

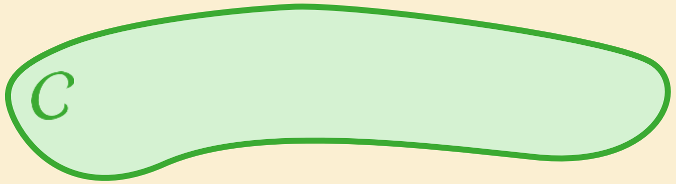
Let $h : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable with regular value 0. The set

$$C = \{\boldsymbol{x} \in \mathbb{R}^n \mid h(\boldsymbol{x}) \geq 0\},$$

is **control invariant** if and only if

$\forall \boldsymbol{x} \in \partial C,\ \exists \boldsymbol{u} \in \mathcal{U},\ L_f h(\boldsymbol{x}) + L_g h(\boldsymbol{x})\boldsymbol{u} \geq 0.$

Definition: *Control Invariance*



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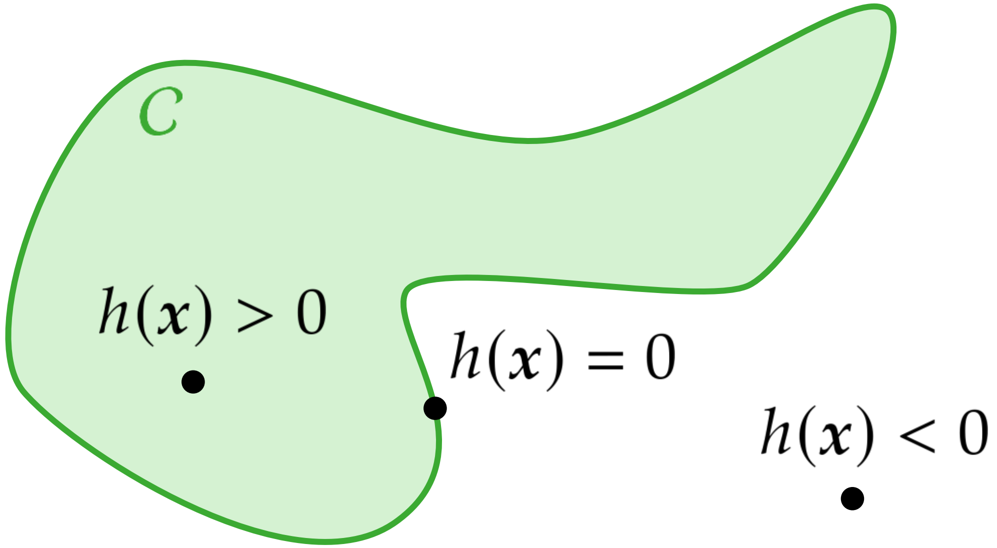

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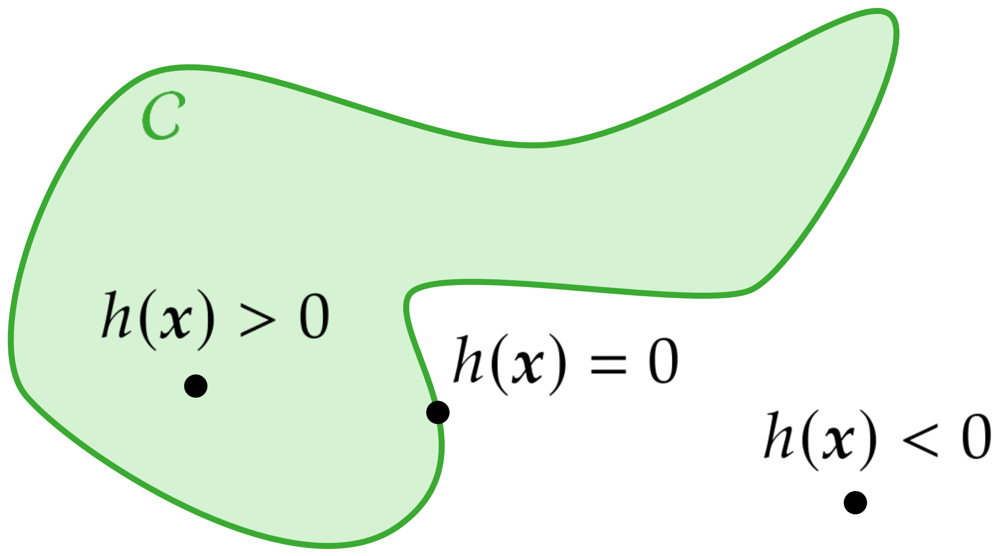

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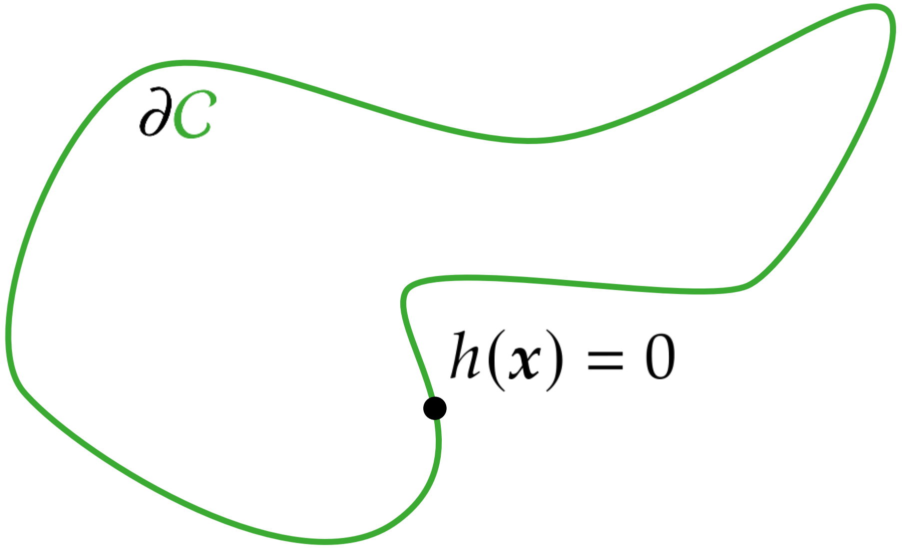

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Theorem: *Nagumo (simplified)*

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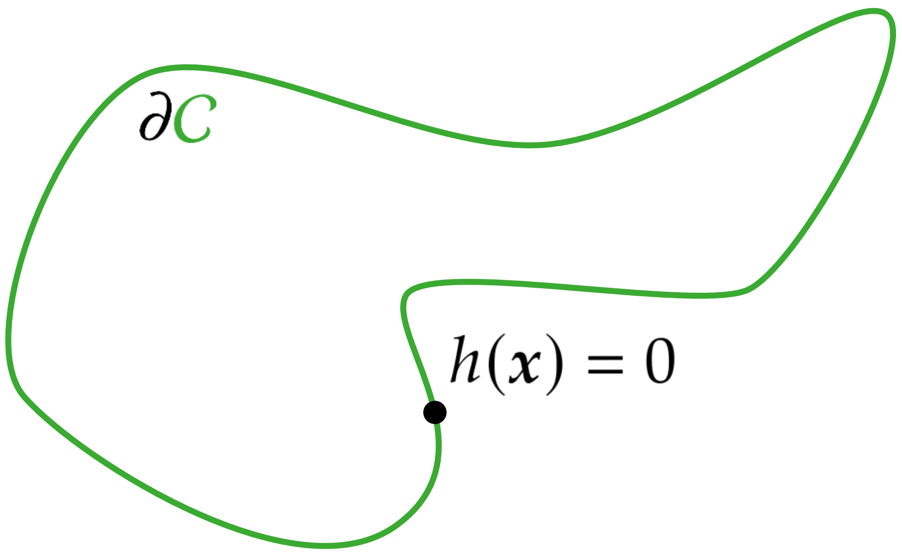

$L_f h(\boldsymbol{x}) + L_g h(\boldsymbol{x})\boldsymbol{u}$

Theorem: *Nagumo (simplified)*

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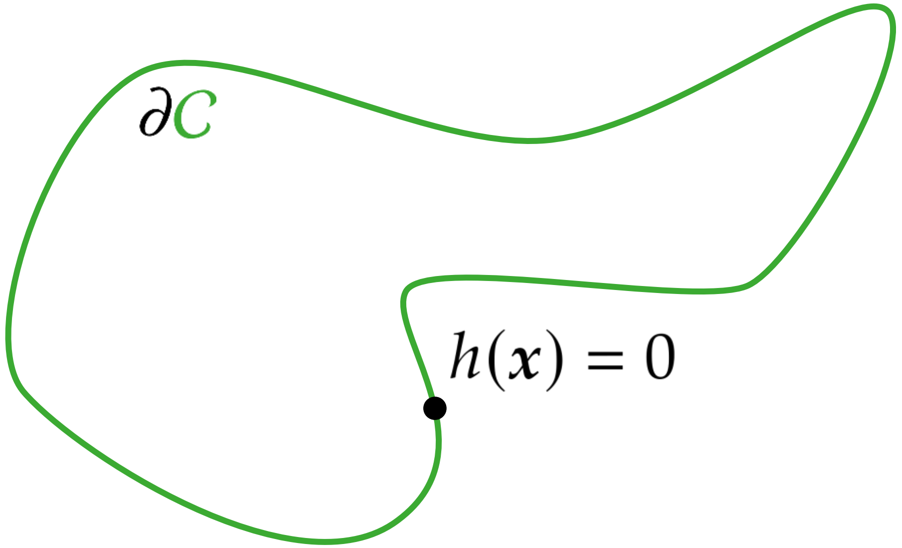

$$L_f h(\boldsymbol{x}) + L_g h(\boldsymbol{x})\boldsymbol{u} = \frac{\partial h}{\partial \boldsymbol{x}} f(\boldsymbol{x}) + \frac{\partial h}{\partial \boldsymbol{x}} g(\boldsymbol{x})\boldsymbol{u}$$

"Lie derivatives" (Notation)

$$L_f h(\boldsymbol{x}) \triangleq \frac{\partial h}{\partial \boldsymbol{x}} f(\boldsymbol{x}) \qquad L_g h(\boldsymbol{x}) \triangleq \frac{\partial h}{\partial \boldsymbol{x}} g(\boldsymbol{x})$$

Nagumo, *Phys.-Math. Soc. Jpn.* 1942
Aubin, Frankowska, **Set-Valued Analysis** 2009

Safety and Invariance

Theorem: *Nagumo (simplified)*

Let $h : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuously differentiable with regular value 0. The set

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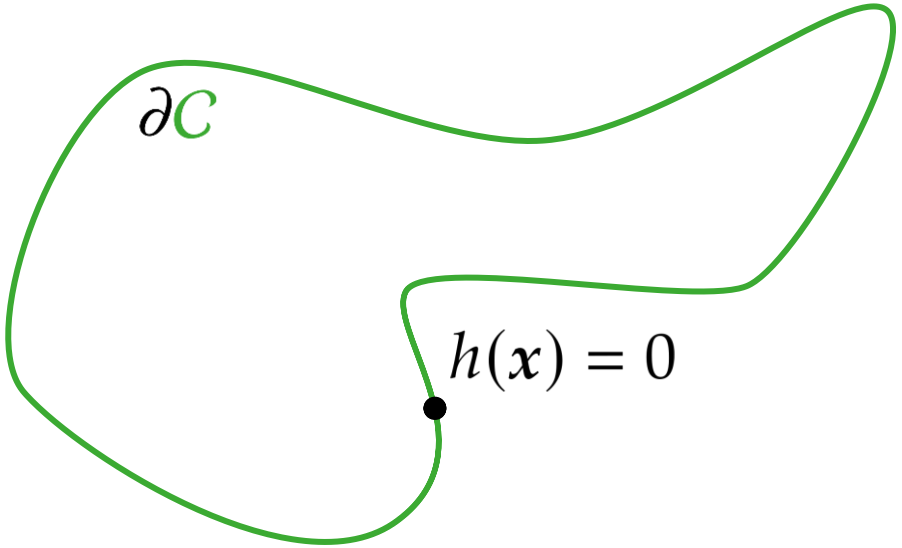

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$$L_f h(\boldsymbol{x}) + L_g h(\boldsymbol{x})\boldsymbol{u} = \frac{\partial h}{\partial \boldsymbol{x}} f(\boldsymbol{x}) + \frac{\partial h}{\partial \boldsymbol{x}} g(\boldsymbol{x})\boldsymbol{u} = \frac{\partial h}{\partial \boldsymbol{x}} \big(f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u}\big)$$

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Aubin, Frankowska, **Set-Valued Analysis** 2009

Safety and Invariance

Theorem: *Nagumo (simplified)*

Let $h : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable with regular value 0. The set

$$C = \{\boldsymbol{x} \in \mathbb{R}^n \mid h(\boldsymbol{x}) \geq 0\},$$

is **control invariant** if and only if

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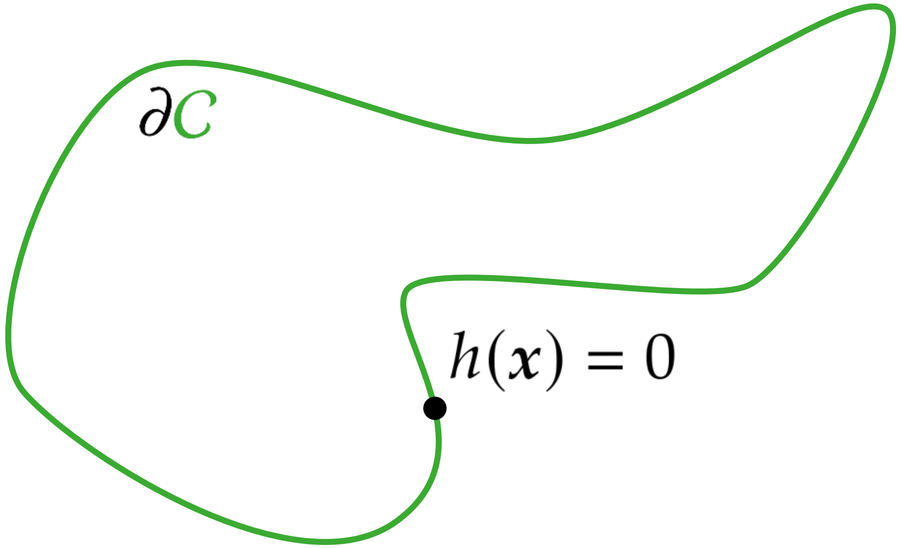

$$L_f h(\boldsymbol{x}) + L_g h(\boldsymbol{x})\boldsymbol{u} = \frac{\partial h}{\partial \boldsymbol{x}} f(\boldsymbol{x}) + \frac{\partial h}{\partial \boldsymbol{x}} g(\boldsymbol{x})\boldsymbol{u}$$
$$= \frac{\partial h}{\partial \boldsymbol{x}}\big(f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u}\big)$$
$$= \frac{\partial h}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}}$$

"Lie derivatives" (Notation)

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Safety and Invariance

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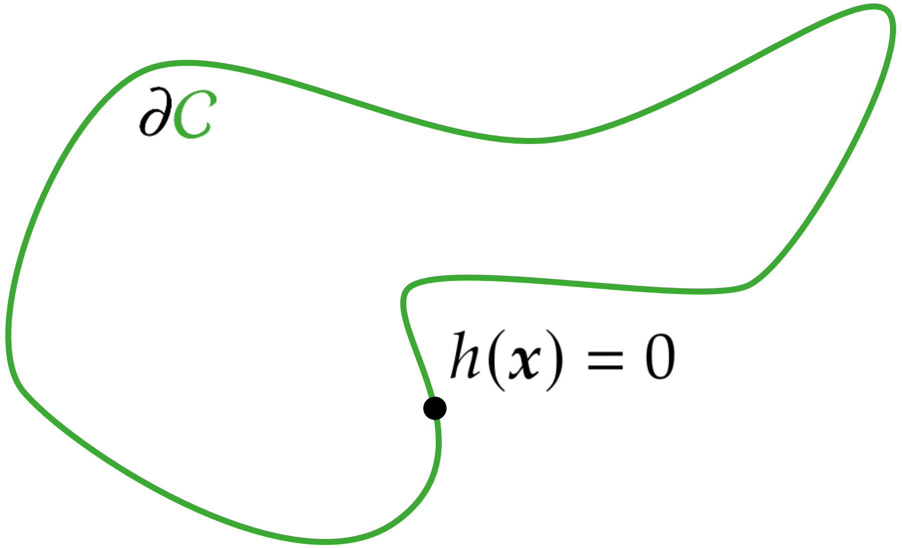

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$$\begin{aligned} L_f h(\boldsymbol{x}) + L_g h(\boldsymbol{x})\boldsymbol{u} &= \frac{\partial h}{\partial \boldsymbol{x}} f(\boldsymbol{x}) + \frac{\partial h}{\partial \boldsymbol{x}} g(\boldsymbol{x})\boldsymbol{u} \\ &= \frac{\partial h}{\partial \boldsymbol{x}} \big(f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u}\big) \\ &= \frac{\partial h}{\partial \boldsymbol{x}} \dot{\boldsymbol{x}} = \dot{h}(\boldsymbol{x}, \boldsymbol{u}) \end{aligned}$$

Safety and Invariance

Theorem: *Nagumo (simplified)*

Let $h : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable with regular value 0. The set

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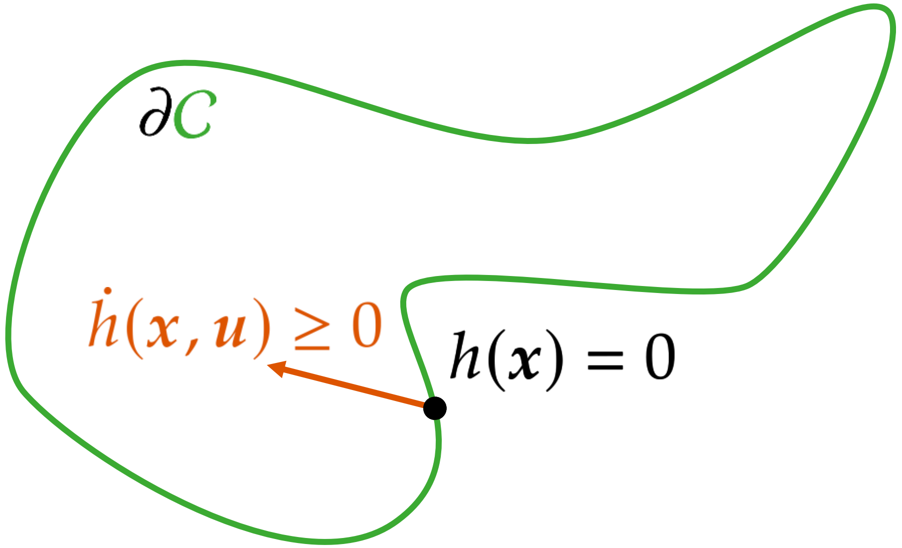

$$L_f h(\boldsymbol{x}) + L_g h(\boldsymbol{x})\boldsymbol{u} = \frac{\partial h}{\partial \boldsymbol{x}} f(\boldsymbol{x}) + \frac{\partial h}{\partial \boldsymbol{x}} g(\boldsymbol{x})\boldsymbol{u} = \frac{\partial h}{\partial \boldsymbol{x}}\left(f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u}\right) = \frac{\partial h}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} = \dot{h}(\boldsymbol{x}, \boldsymbol{u})$$

"Lie derivatives" (Notation)

$$L_f h(\boldsymbol{x}) \triangleq \frac{\partial h}{\partial \boldsymbol{x}} f(\boldsymbol{x}) \qquad L_g h(\boldsymbol{x}) \triangleq \frac{\partial h}{\partial \boldsymbol{x}} g(\boldsymbol{x})$$

Nagumo, *Phys.-Math. Soc. Jpn.* 1942

Aubin, Frankowska, **Set-Valued Analysis** 2009

Safety and Invariance

Theorem: *Nagumo (simplified)*

Let $h : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable with regular value 0. The set

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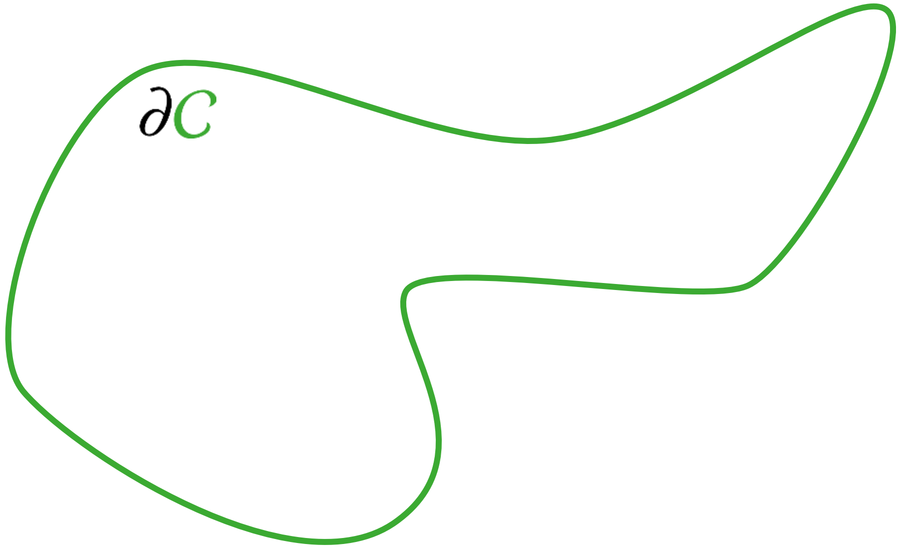
Measure-zero.
Difficult to check numerically.



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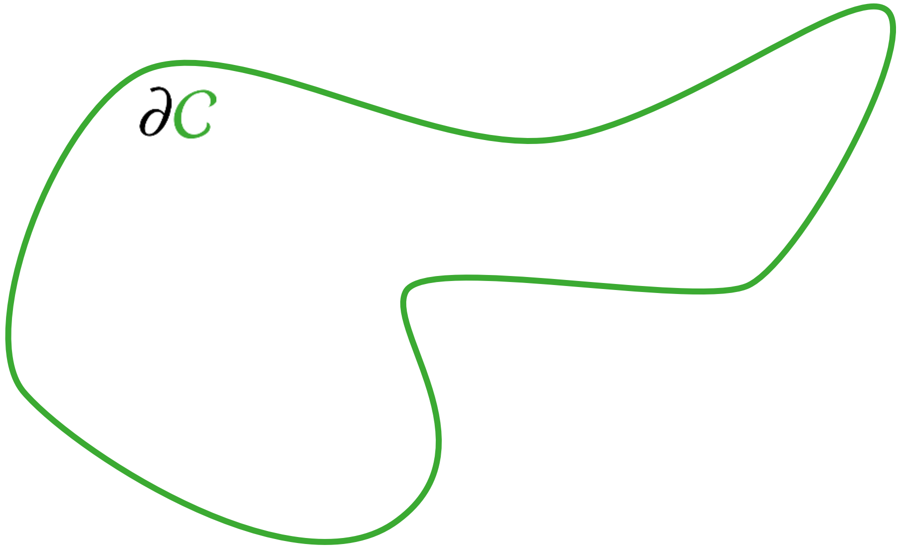
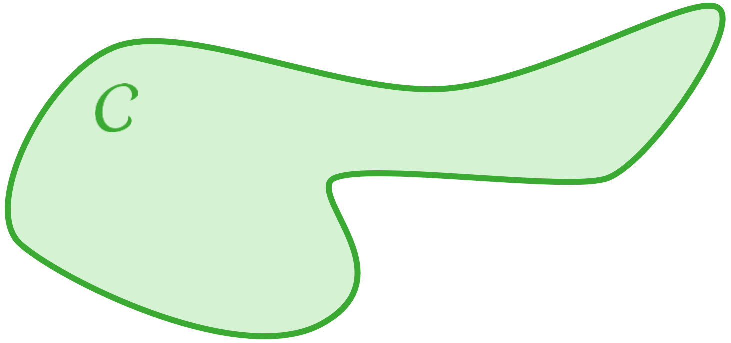


Theorem: *CBF Condition (2017)*

$$\forall x \in C,\ \exists u \in \mathcal{U},\ L_f h(x) + L_g h(x) u + \alpha\big(h(x)\big) \geq 0.$$



Index

Control Barrier Functions (CBFs)

Definition: *CBF*

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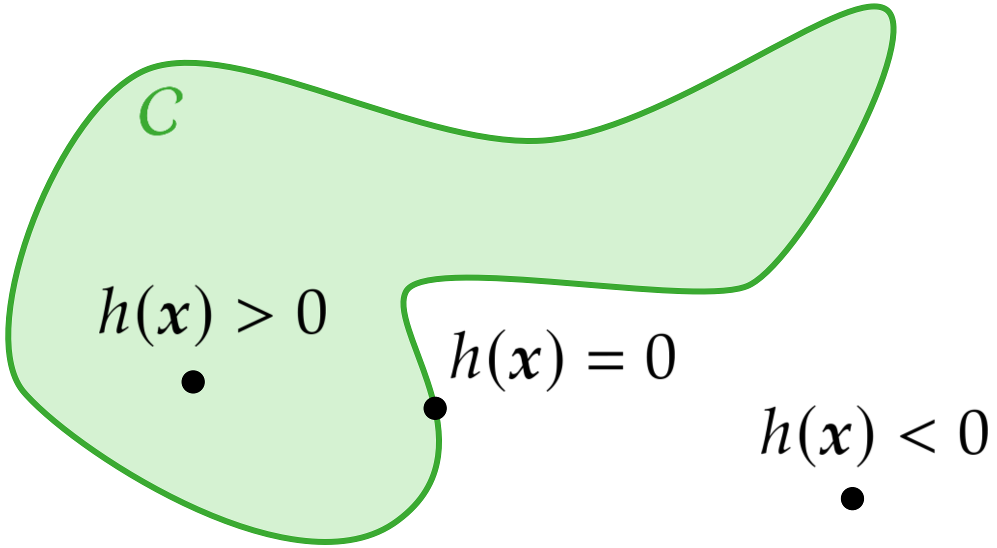

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Ames, Xu, Grizzle, Tabuada, *TAC* 2017

Ames, Coogan, Egerstedt, Notomista, Sreenath, Tabuada, *ECC* 2019 (survey)

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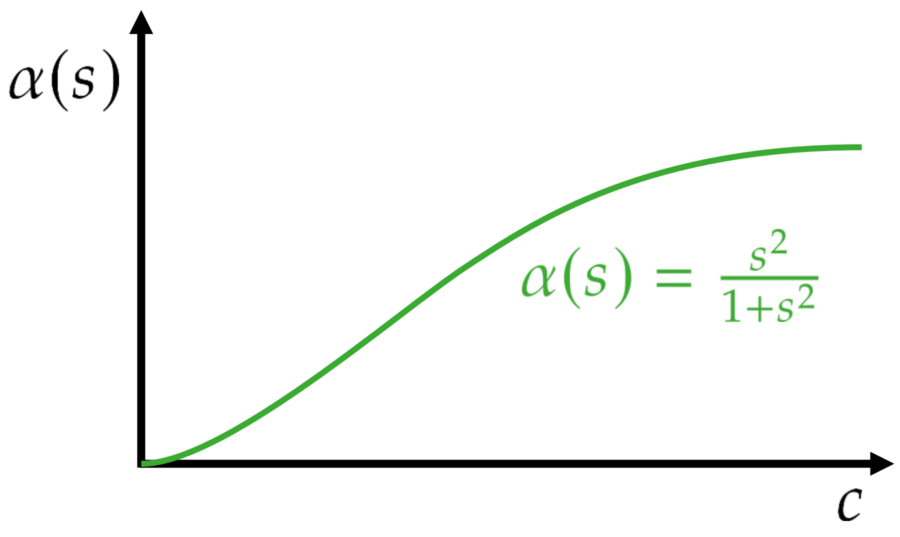

$\alpha \in \mathcal{K}$ if

- continuous
- strictly increasing
- $\alpha(0) = 0$

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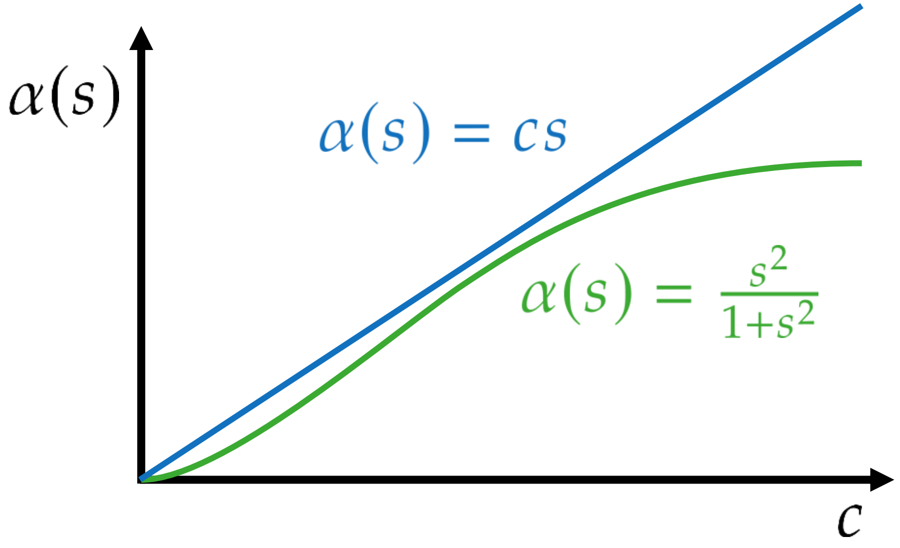

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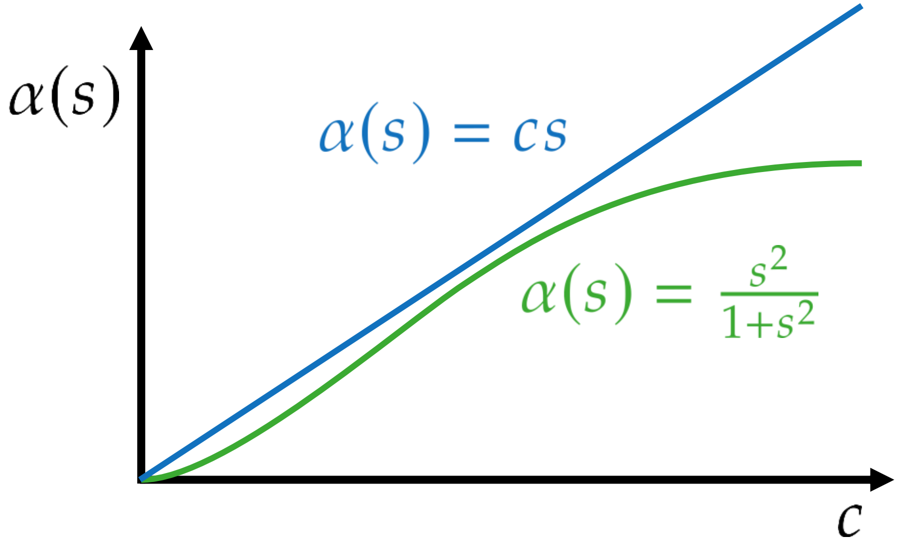

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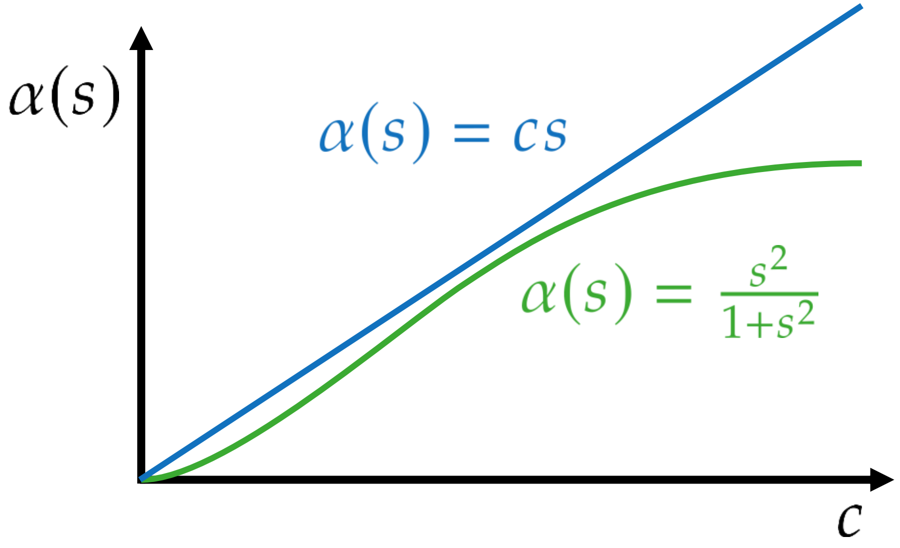

$\boldsymbol{x} \in \partial C$

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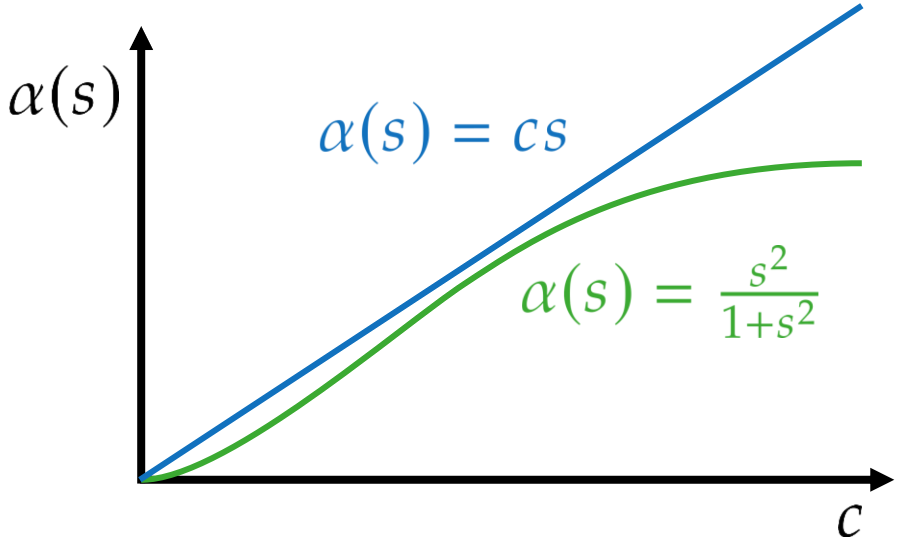

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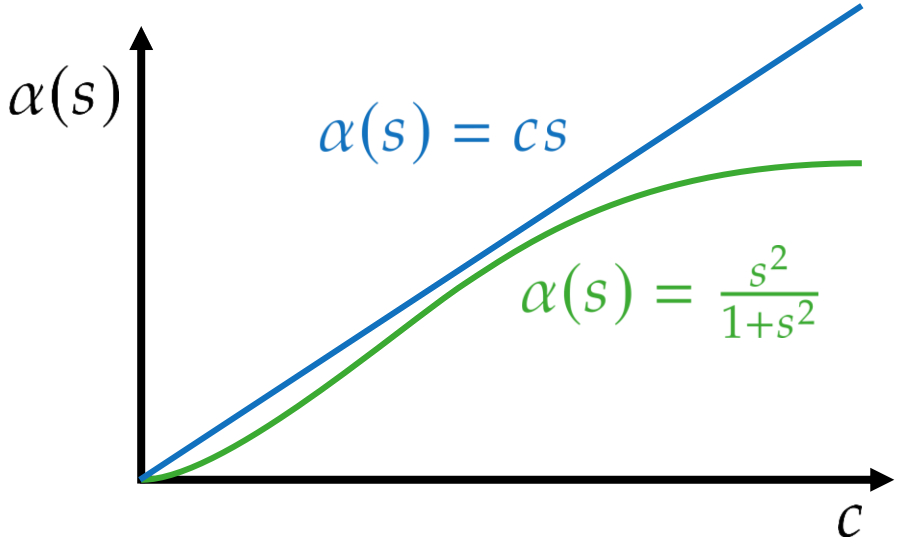

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Theorem: *CBF Condition (2017)*

If h is a CBF, then C is control invariant.

What about the other direction?
If C is control invariant, then h is a CBF.

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Theorem: *CBF Condition (2017)*

If h is a CBF, then C is control invariant.

Theorem: *CBF Condition (2017)*

Assume (a) C is compact and (b) there exists a continuous selection $\mu : C \to \mathcal{U}$ such that $\mu(x) \in \mathcal{K}(x)$.
If C is control invariant, then h is a CBF.

$$\mathcal{K}(x) = \{u \in \mathcal{U} \mid L_f h(x) + L_g h(x) u \geq 0\}$$

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Problem: Constrained Control

Given a control-affine system
$$\dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)u(t)$$
and constraints
$$\mathcal{X} \subset \mathbb{R}^n,\ \mathcal{U} \subset \mathbb{R}^m,$$
find a signal $u : [0, \infty) \to \mathcal{U}$
such that $x(t) \in \mathcal{X}$ for all $t \geq 0$.

Solution exists if there exists $\mathcal{C}$ **control invariant** and **admissible** (i.e., **safe**).

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Given a CBF h, how to find a solution $u(t)$?

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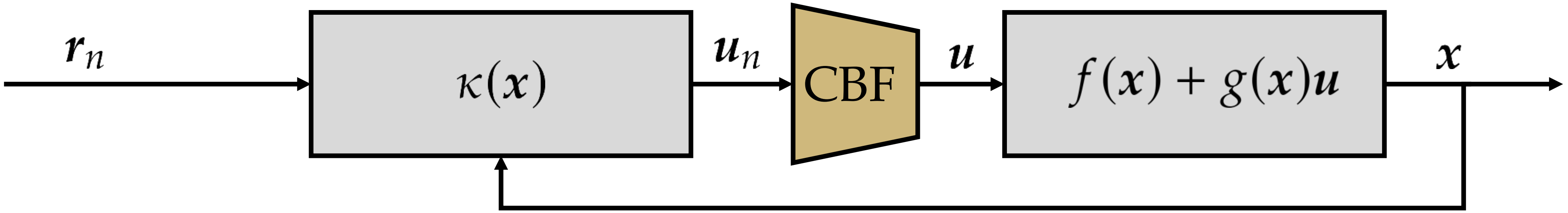
Safety Filters

Family of constrained controllers that use a two-step approach:

1. Design a control law for the **Unconstrained** system,
2. Introduce an **Add-on** unit for constraint enforcement.
   - **Control Barrier Function:** Filter nominal controller



CBF-based Safety Filter

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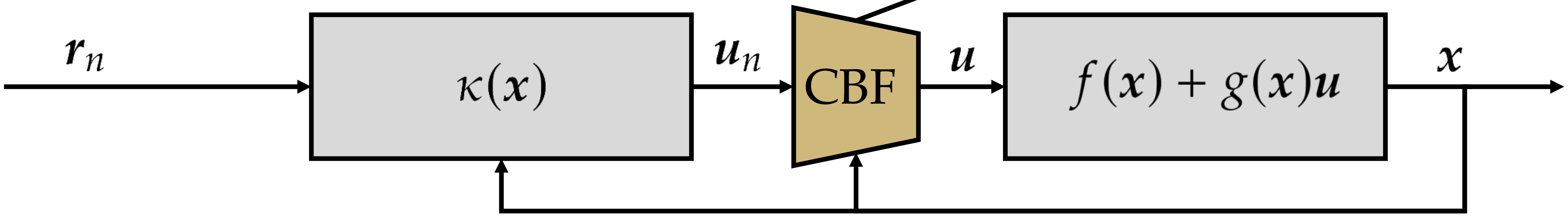
$$\min_{\boldsymbol{u}\in\mathcal{U}} \quad \|\boldsymbol{u} - \kappa(\boldsymbol{x})\|^2$$
$$\text{s.t.} \quad \dot{h}(\boldsymbol{x},\boldsymbol{u}) + \alpha\big(h(\boldsymbol{x})\big) \geq 0$$



CBF-based Safety Filter

$$\begin{aligned} \min_{\boldsymbol{u} \in \mathcal{U}} \quad & \|\boldsymbol{u} - \kappa(\boldsymbol{x})\|^2 \\ \text{s.t.} \quad & L_f h(\boldsymbol{x}) + L_g h(\boldsymbol{x})\boldsymbol{u} \geq -\alpha\big(h(\boldsymbol{x})\big) \end{aligned}$$

CBF-based Safety Filter

Optimization Problem (Program)

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=

Find the vector $\boldsymbol{u} \in \mathcal{U}$ that minimizes $\|\boldsymbol{u} - \kappa(\boldsymbol{x})\|^2$ subject to $L_f h(\boldsymbol{x}) + L_g h(\boldsymbol{x})\boldsymbol{u} \geq -\alpha\big(h(\boldsymbol{x})\big)$.

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Given $\boldsymbol{x}$, these terms are constant.

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Given $\boldsymbol{x}$, these terms are constant.

If $\mathcal{U}$ is polyhedral $\mathcal{U} = \{\boldsymbol{u} \mid M\boldsymbol{u} \leq \boldsymbol{b}\}$,
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Commercial QP Solvers

mosek

GUROBI OPTIMIZATION

COPT Cardinal Optimizer

IBM CPLEX

CBF-based Safety Filter

Optimization Problem

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Is there a solution?

(feasibility)

CBF-based Safety Filter

Optimization Problem

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Theorem: *CBF Safety Filter*

If h is a CBF, then the CBF-based safety filter is feasible for all $\boldsymbol{x} \in \mathcal{C}$.

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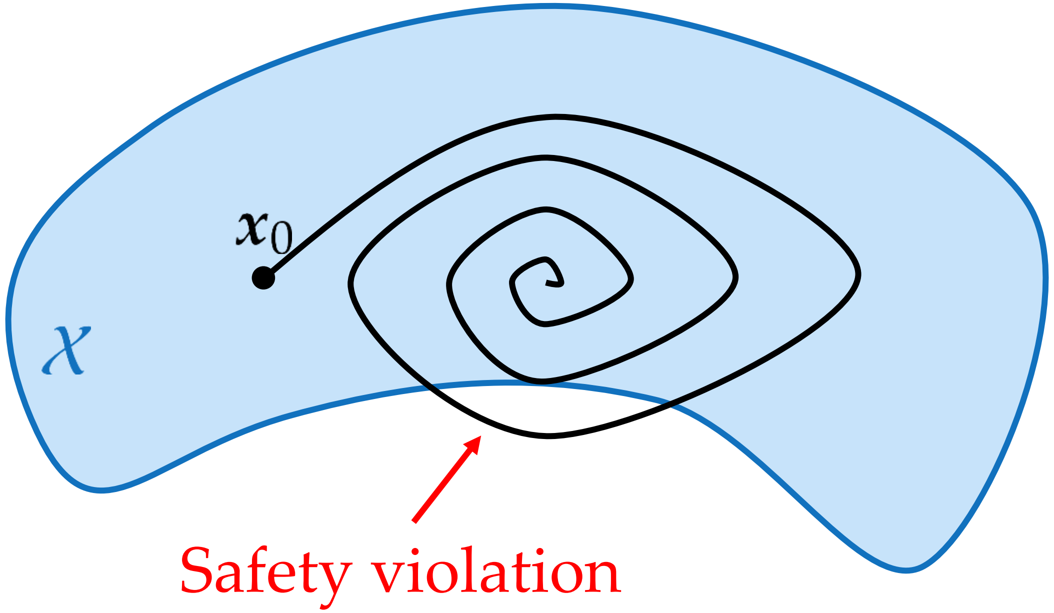
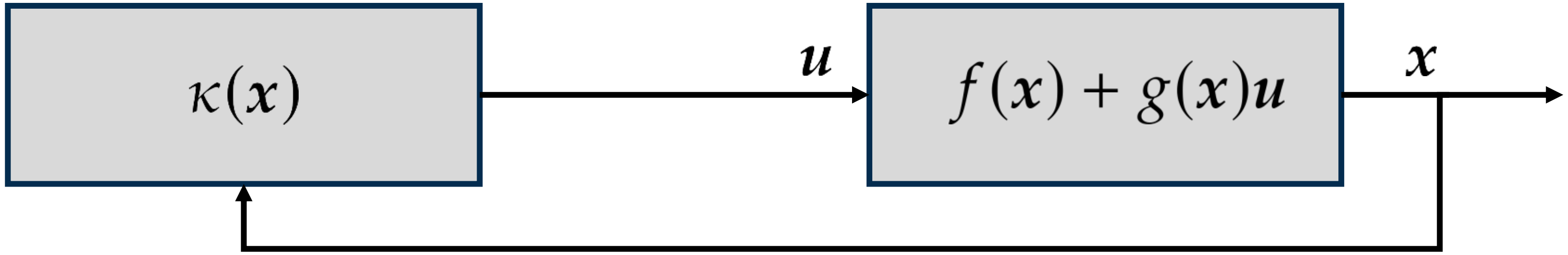


Unsafe



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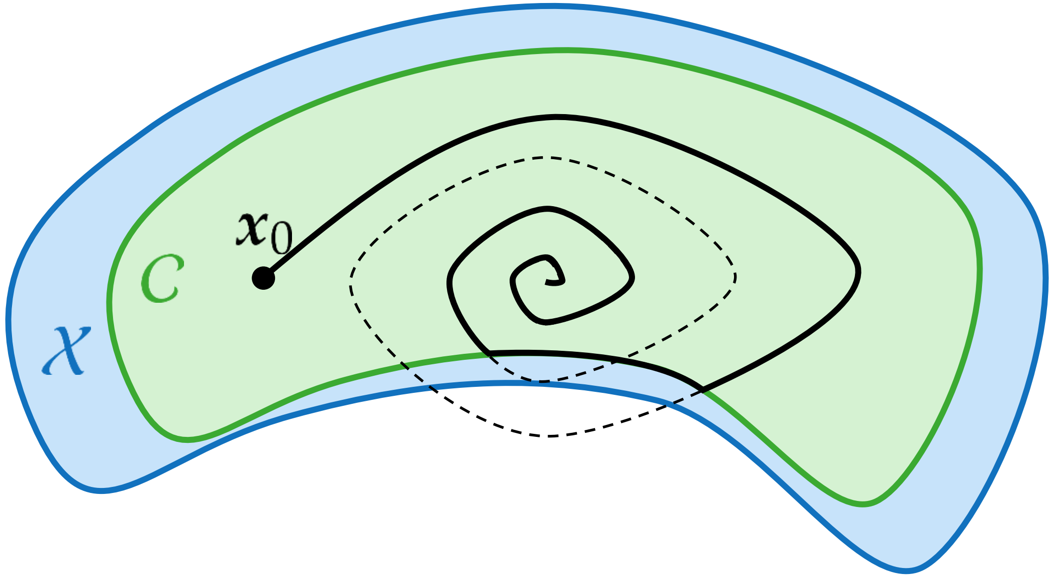
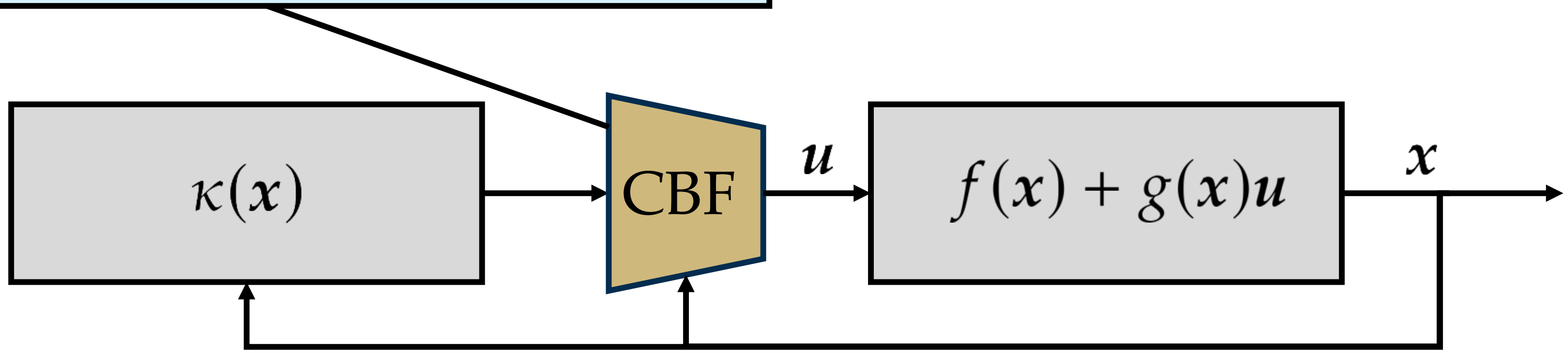


$$\begin{aligned} \min_{\boldsymbol{u}\in\mathcal{U}} \quad & \|\boldsymbol{u} - \kappa(\boldsymbol{x})\|^2 \\ \text{s.t.} \quad & L_f h(\boldsymbol{x}) + L_g h(\boldsymbol{x})\boldsymbol{u} \geq -\alpha\big(h(\boldsymbol{x})\big) \end{aligned}$$

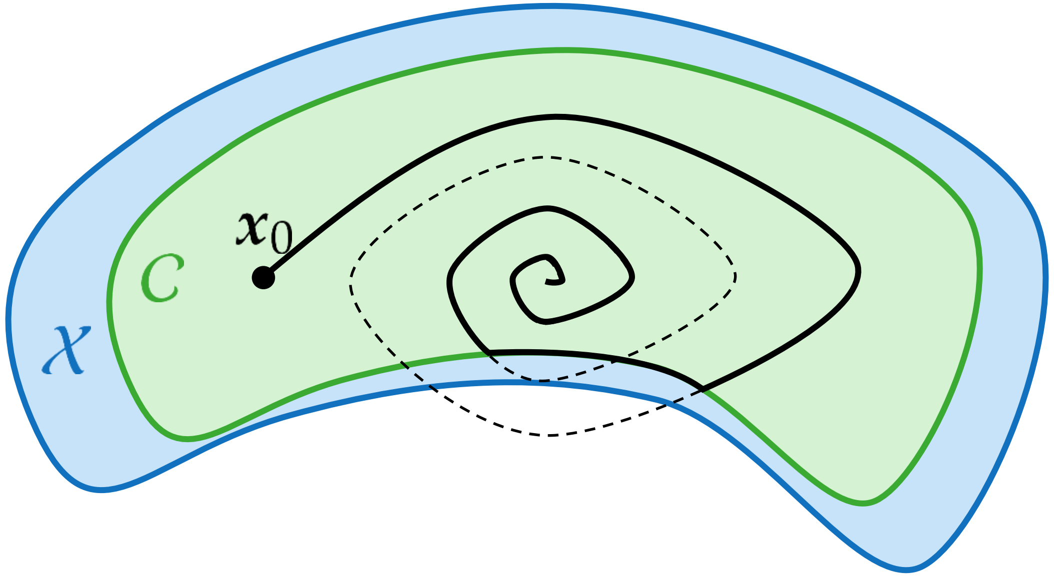
Safe



CBF-based Safety Filter

Safety Filters

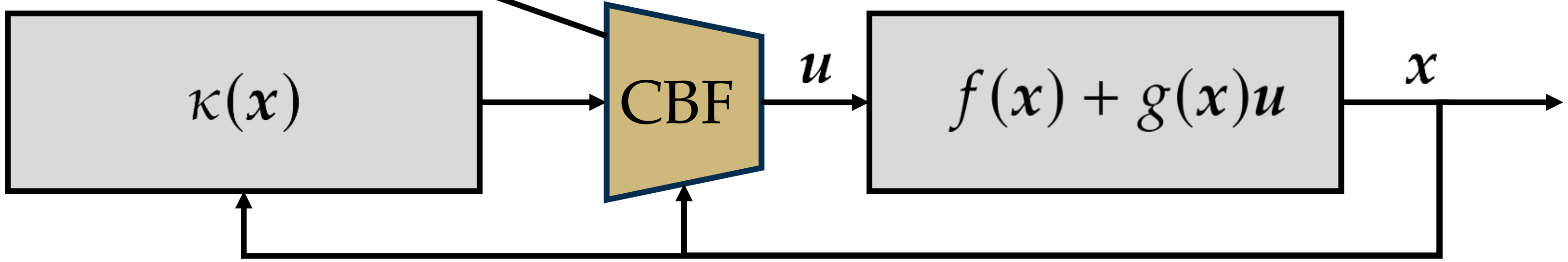
1. Design a control law for the **Unconstrained** system,
2. Introduce an **Add-on** unit for constraint enforcement.



$$\min_{\boldsymbol{u}\in\mathcal{U}} \quad \|\boldsymbol{u} - \kappa(\boldsymbol{x})\|^2$$
$$\text{s.t.} \quad L_f h(\boldsymbol{x}) + L_g h(\boldsymbol{x})\boldsymbol{u} \geq -\alpha\big(h(\boldsymbol{x})\big)$$

How to find a CBF $h(\boldsymbol{x})$?

Safe



Index

Example: Safety Filter for F-16

F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix}\theta & \dot{\theta} & \alpha_w\end{bmatrix}$$

$$\boldsymbol{u} = \begin{bmatrix}\delta_e & \delta_f\end{bmatrix}$$

$$f(\boldsymbol{x}) = A\boldsymbol{x} \quad g(\boldsymbol{x}) = B$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.87 & 43.22 \\ 0 & 0.99 & -1.34 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ -17.25 & -1.58 \\ -0.17 & -0.25 \end{bmatrix}$$

θ pitch $\quad \alpha_w$ angle of attack $\quad \delta_e, \delta_f$ control surfaces

Example: Safety Filter for F-16

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -0.87 & 43.22 & -17.25 & -1.58 \\ 0 & 0.99 & -1.34 & -0.17 & -0.25 \\ 0 & 0 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & -20 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 20 & 0 \\ 0 & 20 \end{bmatrix}$$

Actuator Inner-loop (first-order lag)

F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

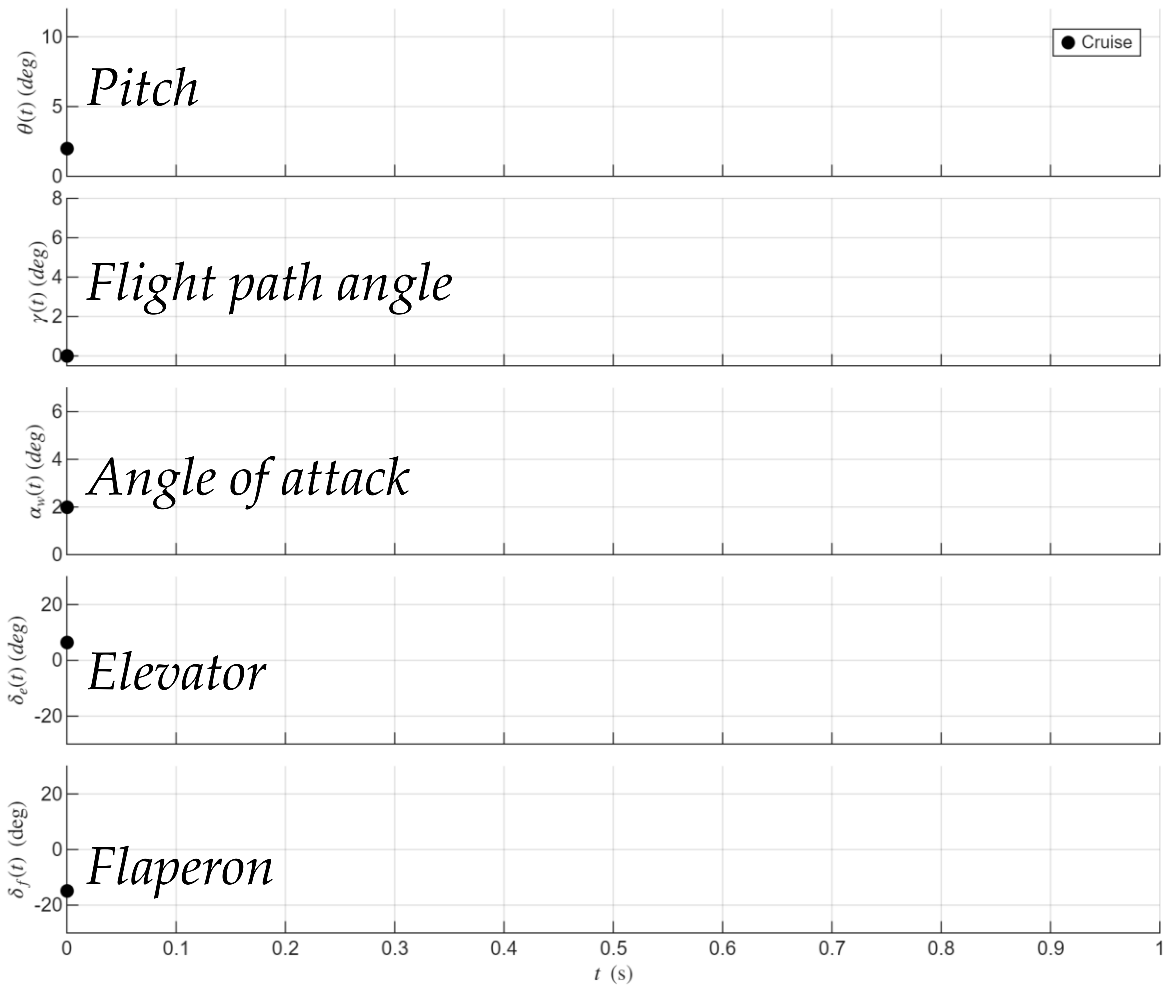
$$\boldsymbol{u} = \begin{bmatrix} \delta_{e,c} & \delta_{f,c} \end{bmatrix}$$

$$f(\boldsymbol{x}) = A\boldsymbol{x} \quad g(\boldsymbol{x}) = B$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.87 & 43.22 \\ 0 & 0.99 & -1.34 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ -17.25 & -1.58 \\ -0.17 & -0.25 \end{bmatrix}$$

θ pitch $\quad \alpha_w$ angle of attack $\quad \delta_e, \delta_f$ control surfaces

Example: Safety Filter for F-16



F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

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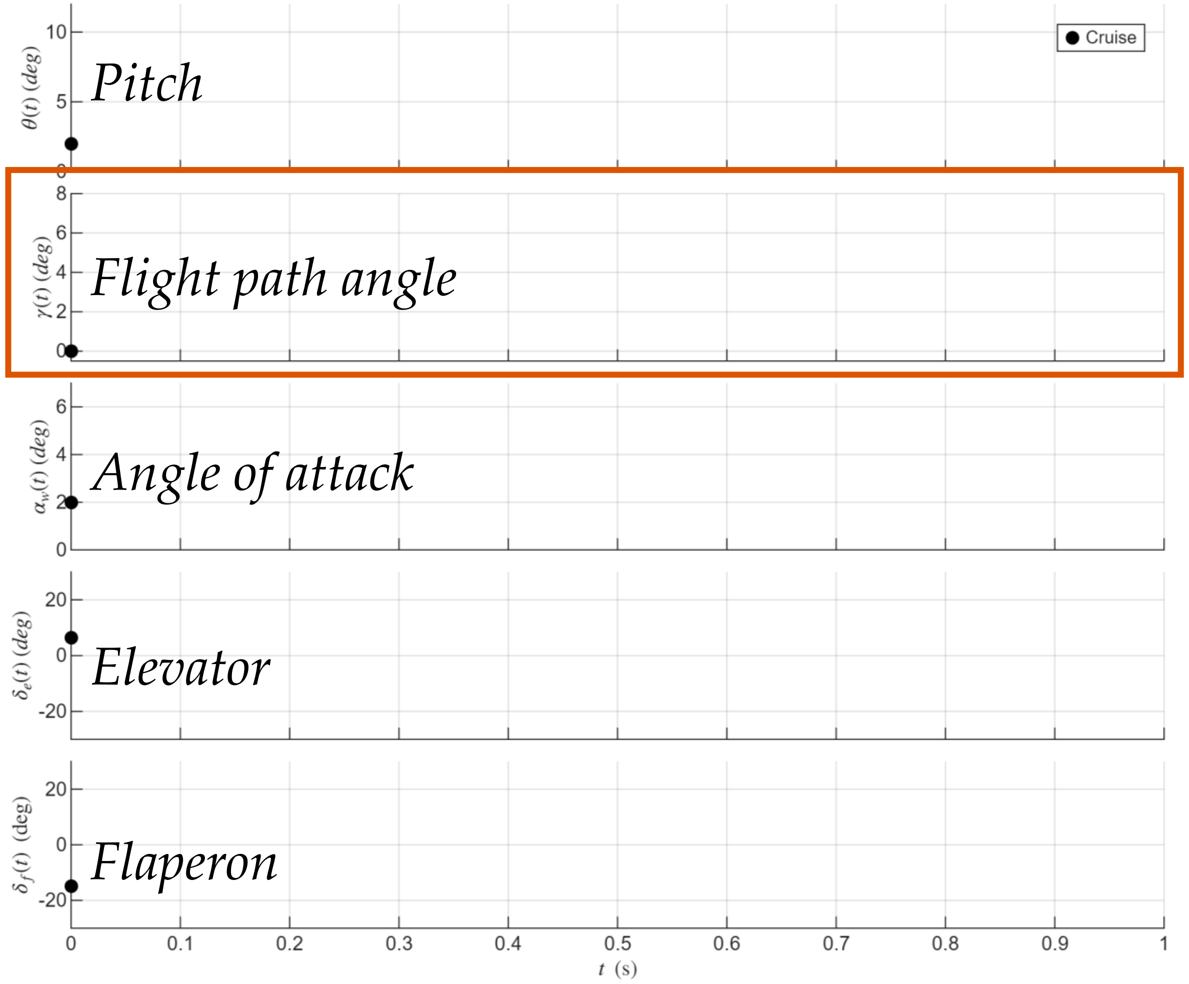
Cruise: $\theta = 2,\ \gamma = 0$

Example: Safety Filter for F-16



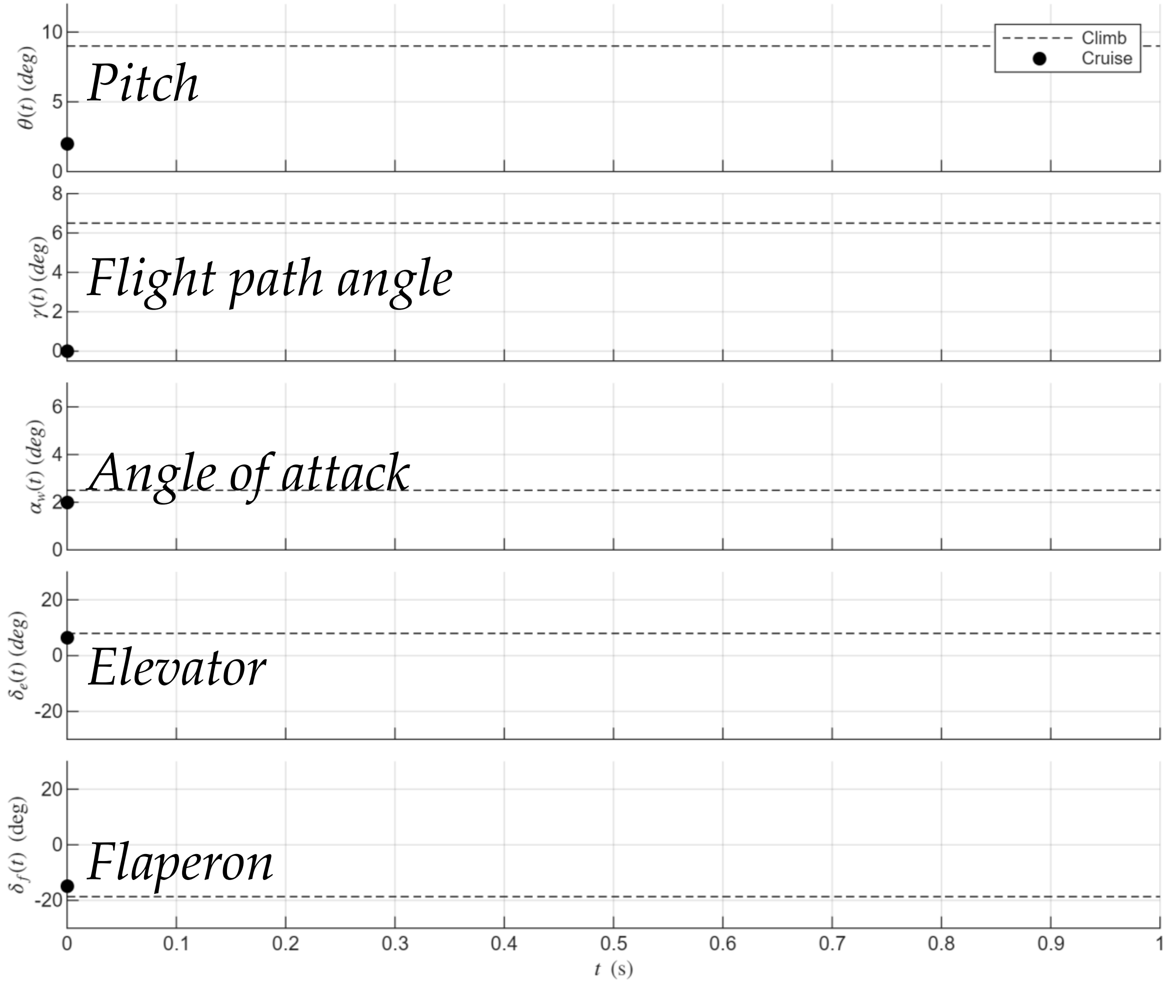
F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

$$\boldsymbol{u} = \begin{bmatrix} \delta_{e,c} & \delta_{f,c} \end{bmatrix}$$

Cruise: $\theta = 2$, $\gamma = 0$ $\quad \gamma \approx \theta - \alpha_w$

Example: Safety Filter for F-16



F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

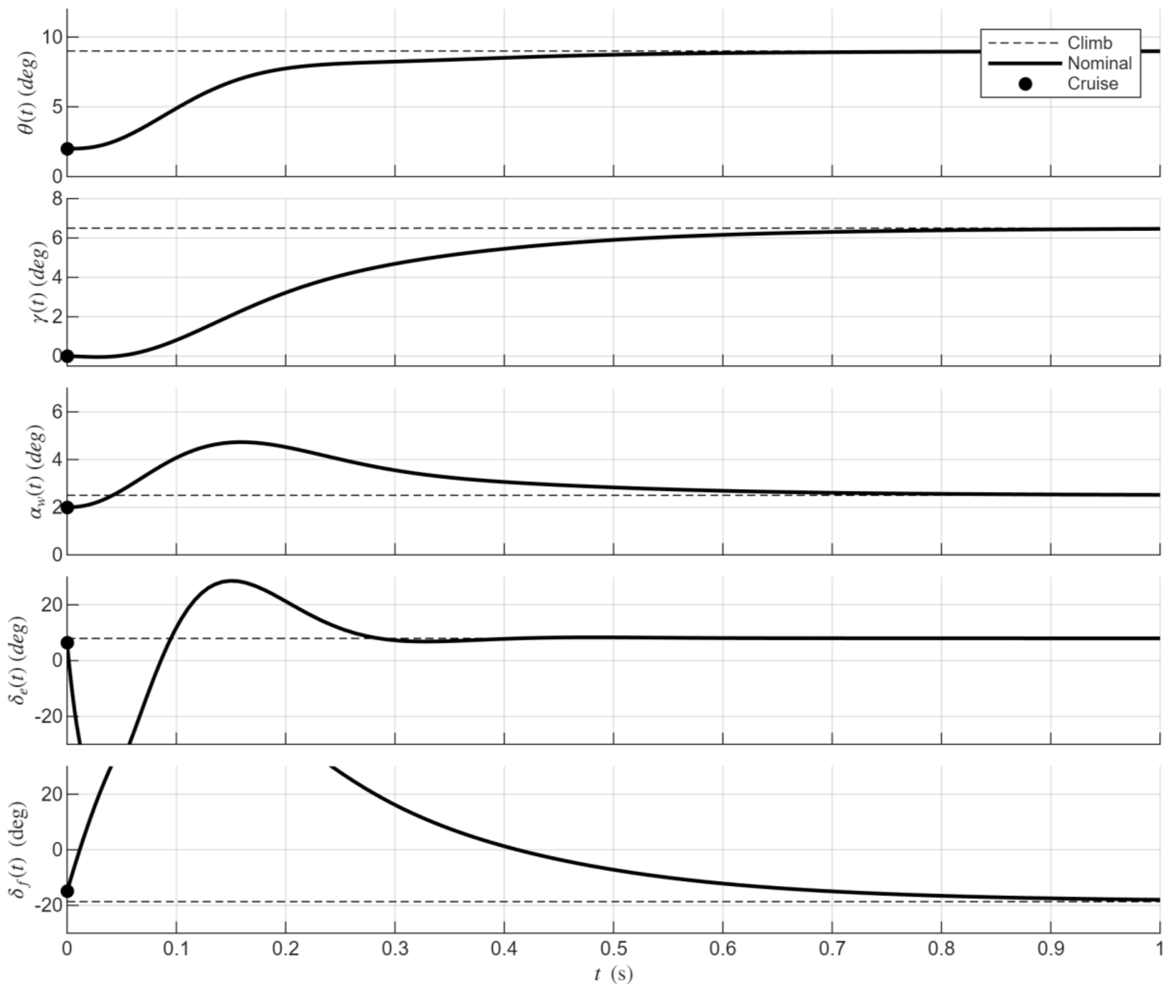
$$\boldsymbol{u} = \begin{bmatrix} \delta_{e,c} & \delta_{f,c} \end{bmatrix}$$

Cruise: $\theta = 2,\ \gamma = 0$

↓ Goal

Climb: $\theta = 9,\ \gamma = 6.5$

Example: Safety Filter for F-16



F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

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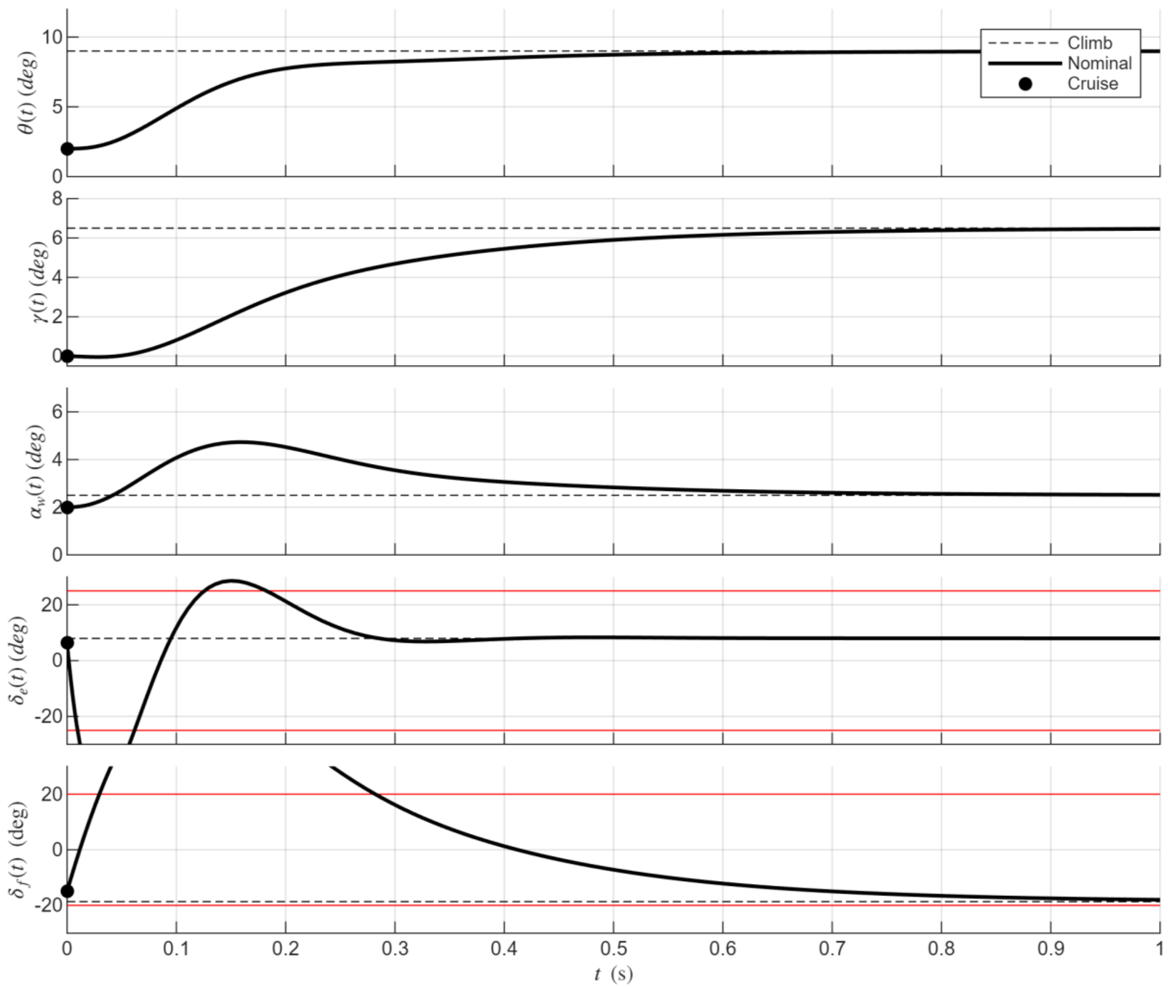
LQR Controller $\kappa(\boldsymbol{x}) = \bar{u}(\boldsymbol{r}) - K(\boldsymbol{x} - \bar{x}(\boldsymbol{r}))$

Desired **Climb** operating conditions

Example: Safety Filter for F-16



F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

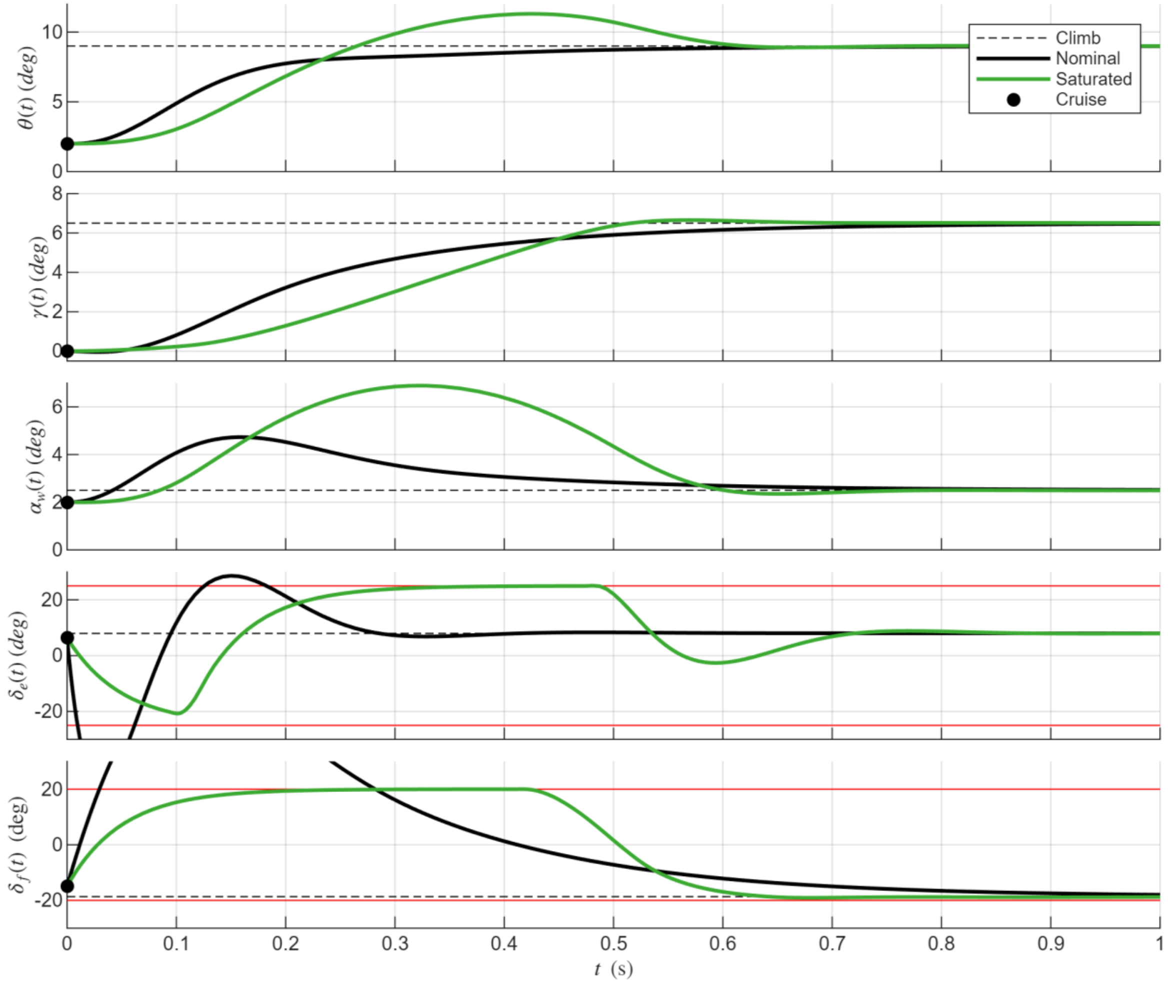
$$\boldsymbol{u} = \begin{bmatrix} \delta_{e,c} & \delta_{f,c} \end{bmatrix}$$

LQR Controller $\kappa(\boldsymbol{x}) = \bar{u}(\boldsymbol{r}) - K(\boldsymbol{x} - \bar{x}(\boldsymbol{r}))$

Control surface deflection limits

$|\delta_e| \leq 25$ deg, $|\delta_f| \leq 20$ deg

Example: Safety Filter for F-16



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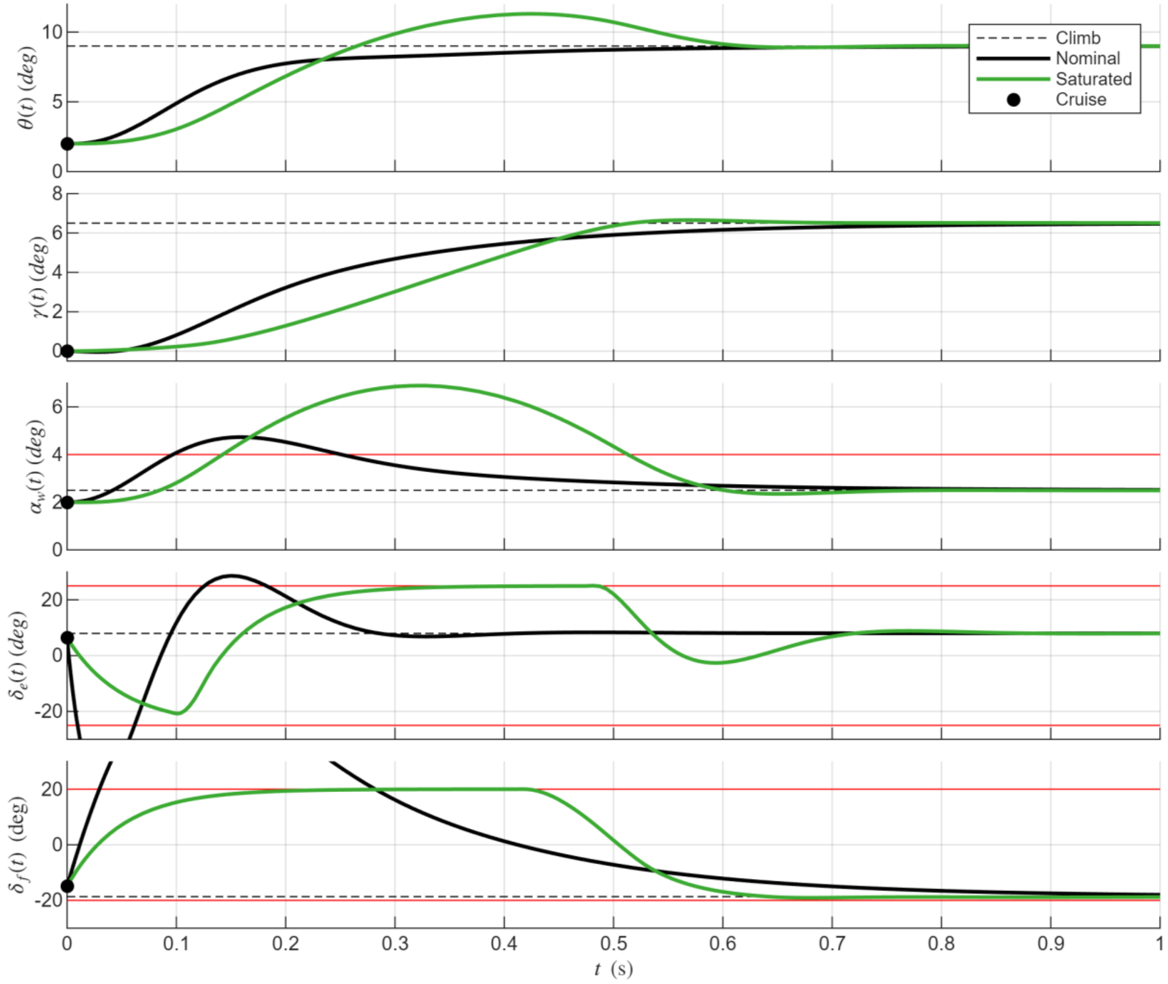
LQR Controller $\kappa(\boldsymbol{x}) = \bar{u}(\boldsymbol{r}) - K(\boldsymbol{x} - \bar{x}(\boldsymbol{r}))$

Saturated LQR $\boldsymbol{u} = \mathrm{sat}(\kappa(\boldsymbol{x}))$

Control surface deflection limits

$|\delta_e| \leq 25$ deg, $|\delta_f| \leq 20$ deg

Example: Safety Filter for F-16



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Saturated LQR $\boldsymbol{u} = \mathrm{sat}\big(\kappa(\boldsymbol{x})\big)$

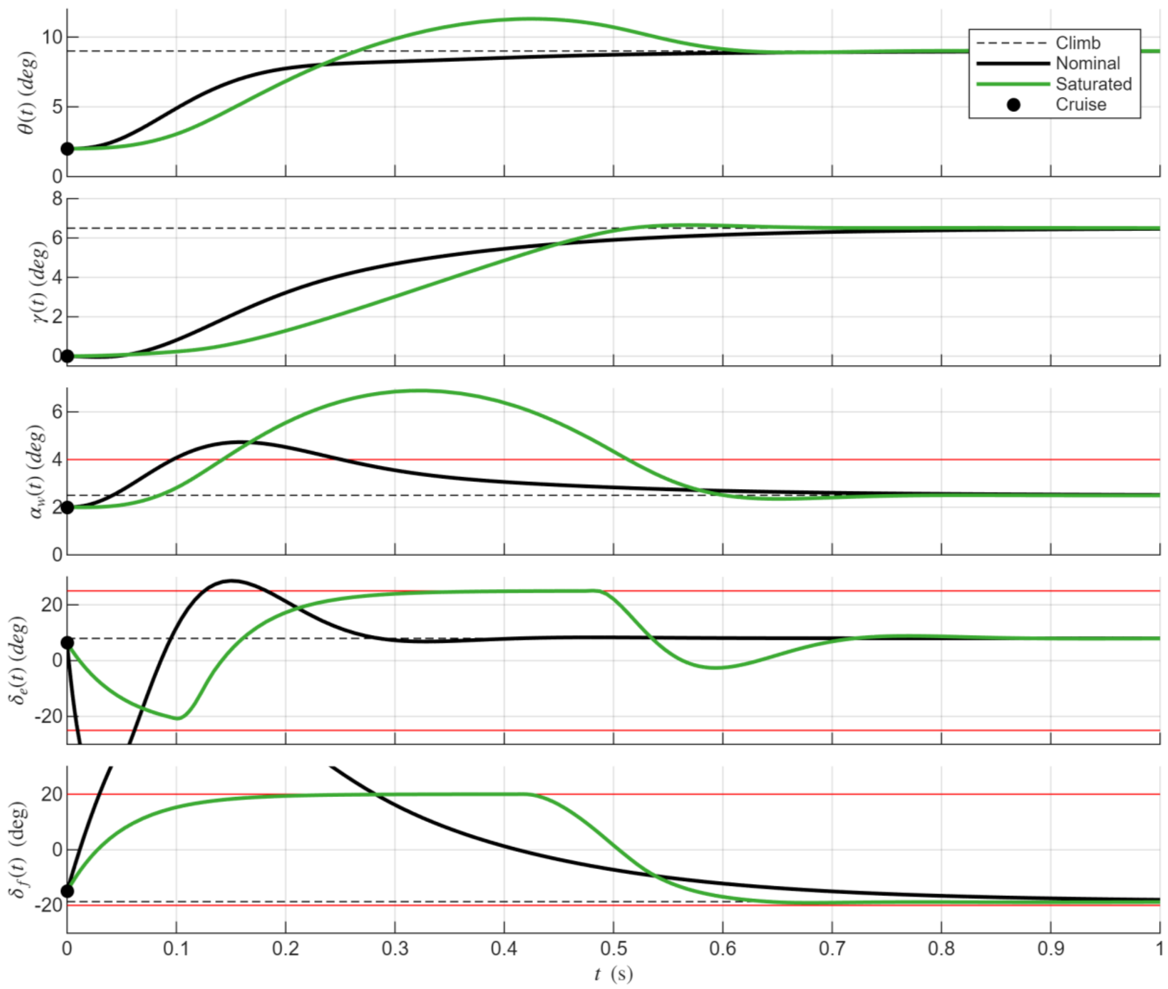
Pitch pointing constraint

$|\alpha_w| \leq 25$ deg

Control surface deflection limits

$|\delta_e| \leq 25$ deg, $|\delta_f| \leq 20$ deg

Example: Safety Filter for F-16



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LQR Controller $\kappa(\boldsymbol{x}) = \bar{u}(\boldsymbol{r}) - K(\boldsymbol{x} - \bar{x}(\boldsymbol{r}))$

Saturated LQR $\boldsymbol{u} = \mathrm{sat}(\kappa(\boldsymbol{x}))$

Saturation is not good enough.
Let's try CBFs

Example: Safety Filter for F-16

Naïve CBF Design (elevator)

This procedure yields ***candidate CBFs***.
They provide **no safety guarantees**.

F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

$$\boldsymbol{u} = \begin{bmatrix} \delta_{e,c} & \delta_{f,c} \end{bmatrix}$$

$$25 \text{ deg} \geq \delta_\mathrm{e}$$

Example: Safety Filter for F-16

Naïve CBF Design (elevator)

Elevator constraint: $h(\boldsymbol{x}) = 25 - \delta_e \geq 0$

This procedure yields ***candidate CBFs***.
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F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

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Example: Safety Filter for F-16

Naïve CBF Design (elevator)

Elevator constraint: $h(\boldsymbol{x}) = 25 - \delta_e \geq 0$

Differentiate: $\frac{\partial h}{\partial x} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \end{bmatrix}$

⚠ This procedure yields ***candidate CBFs***. They provide **no safety guarantees**.

F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

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Elevator constraint: $h(\boldsymbol{x}) = 25 - \delta_e \geq 0$

Differentiate: $\frac{\partial h}{\partial x} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \end{bmatrix}$

Compute: $\dot{h}(\boldsymbol{x}, \boldsymbol{u}) = L_f h(\boldsymbol{x}) + L_g h(\boldsymbol{x})\boldsymbol{u}$

$$L_f h(\boldsymbol{x}) = \frac{\partial h}{\partial \boldsymbol{x}} A\boldsymbol{x}$$

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F-16 Pitch Dynamics

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$\boldsymbol{u} = \begin{bmatrix} \delta_{e,c} & \delta_{f,c} \end{bmatrix}$

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$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -0.87 & 43.22 & -17.25 & -1.58 \\ 0 & 0.99 & -1.34 & -0.17 & -0.25 \\ 0 & 0 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & -20 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 20 & 0 \\ 0 & 20 \end{bmatrix}$$

Example: Safety Filter for F-16

Naïve CBF Design (elevator)

Elevator constraint: $h(\boldsymbol{x}) = 25 - \delta_e \geq 0$

Differentiate: $\frac{\partial h}{\partial x} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \end{bmatrix}$

Compute: $\dot{h}(\boldsymbol{x}, \boldsymbol{u}) = L_f h(\boldsymbol{x}) + L_g h(\boldsymbol{x})\boldsymbol{u}$

$$L_f h(\boldsymbol{x}) = \frac{\partial h}{\partial \boldsymbol{x}} A\boldsymbol{x} = 20\delta_e$$

$$L_g h(\boldsymbol{x})\boldsymbol{u} = \frac{\partial h}{\partial \boldsymbol{x}} B\boldsymbol{u} = -20\delta_{e,c}$$

⚠ This procedure yields ***candidate CBFs***.
They provide **no safety guarantees**.

F-16 Pitch Dynamics

$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$

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Example: Safety Filter for F-16

Naïve CBF Design (elevator)

Elevator constraint: $h(\boldsymbol{x}) = 25 - \delta_e \geq 0$

Differentiate: $\frac{\partial h}{\partial x} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \end{bmatrix}$

Compute: $\dot{h}(\boldsymbol{x}, \boldsymbol{u}) = 20(\delta_e - \delta_{e,c})$

This procedure yields ***candidate CBFs***.
They provide **no safety guarantees**.

F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

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$$f(\boldsymbol{x}) = A\boldsymbol{x} \quad g(\boldsymbol{x}) = B$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -0.87 & 43.22 & -17.25 & -1.58 \\ 0 & 0.99 & -1.34 & -0.17 & -0.25 \\ 0 & 0 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & -20 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 20 & 0 \\ 0 & 20 \end{bmatrix}$$

Example: Safety Filter for F-16

Naïve CBF Design (elevator)

Elevator constraint: $h(\boldsymbol{x}) = 25 - \delta_e \geq 0$

Differentiate: $\frac{\partial h}{\partial \boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \end{bmatrix}$

Compute: $\dot{h}(\boldsymbol{x}, \boldsymbol{u}) = 20(\delta_e - \delta_{e,c})$

Enforce: $\dot{h}(\boldsymbol{x}, \boldsymbol{u}) \geq -\alpha\big(h(\boldsymbol{x})\big)$

This procedure yields ***candidate CBFs***.
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F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

$$\boldsymbol{u} = \begin{bmatrix} \delta_{e,c} & \delta_{f,c} \end{bmatrix}$$

$$\begin{aligned} \min_{\boldsymbol{u} \in \mathbb{R}^2} \quad & \|\boldsymbol{u} - \kappa(\boldsymbol{x})\|^2 \\ \text{s.t.} \quad & \underbrace{20(\delta_e - \delta_{e,c})}_{\dot{h}(\boldsymbol{x}, \boldsymbol{u})} \geq -\alpha \underbrace{(25 - \delta_e)}_{h(\boldsymbol{x})} \end{aligned}$$

Example: Safety Filter for F-16

Naïve CBF Design (elevator)

Elevator constraint: $h(\boldsymbol{x}) = 25 - \delta_e \geq 0$

Differentiate: $\frac{\partial h}{\partial x} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \end{bmatrix}$

Compute: $\dot{h}(\boldsymbol{x}, \boldsymbol{u}) = 20(\delta_e - \delta_{e,c})$

Enforce: $\dot{h}(\boldsymbol{x}, \boldsymbol{u}) \geq -\alpha\big(h(\boldsymbol{x})\big)$

Tune: $\alpha > 0$

Safety ------------------------------ Performance

$\alpha = 0.1$ $\alpha = 100$

This procedure yields ***candidate CBFs***.
They provide **no safety guarantees**.

F-16 Pitch Dynamics

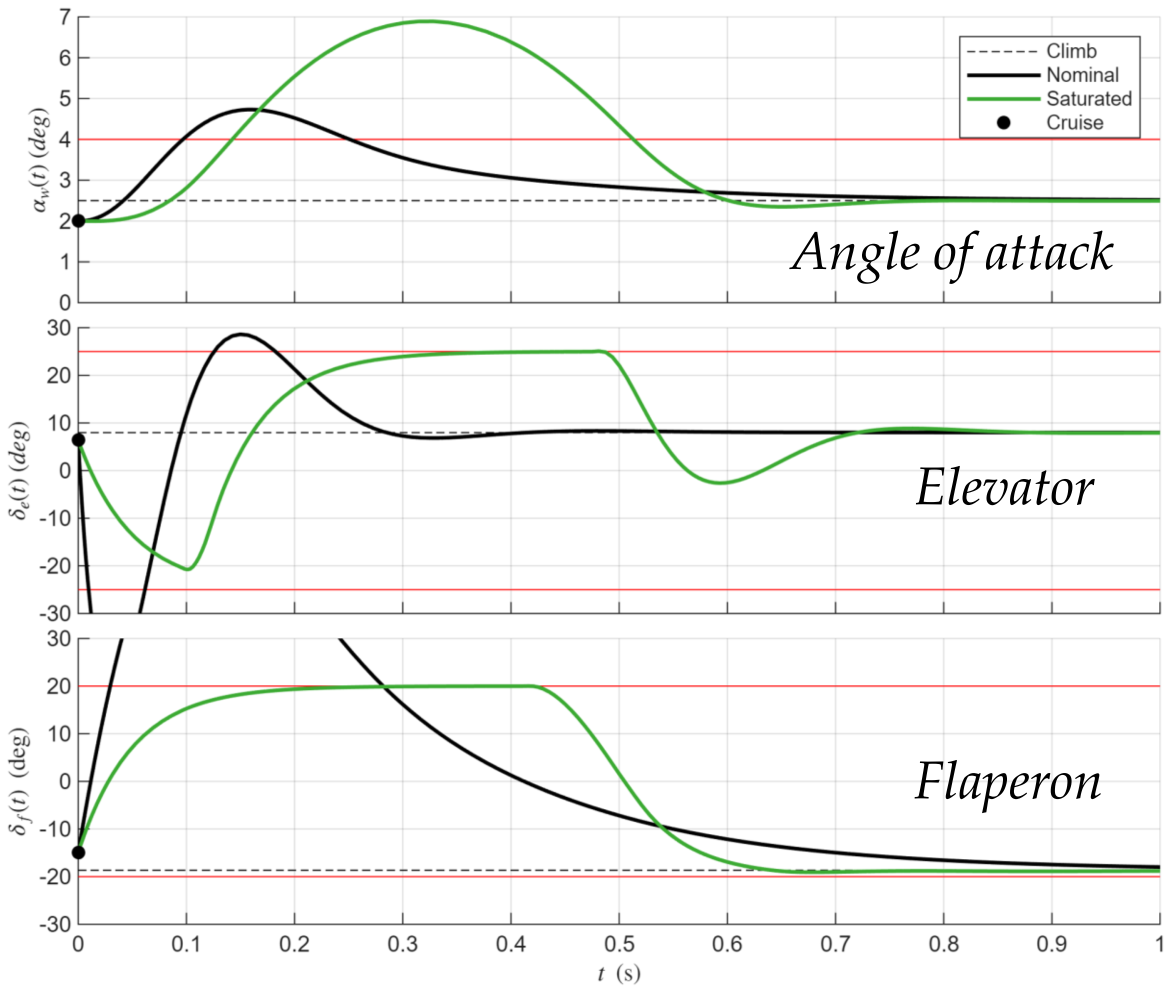
$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

$$\boldsymbol{u} = \begin{bmatrix} \delta_{e,c} & \delta_{f,c} \end{bmatrix}$$

$$\min_{\boldsymbol{u} \in \mathbb{R}^2} \quad \|\boldsymbol{u} - \kappa(\boldsymbol{x})\|^2$$

$$\text{s.t.} \quad \underbrace{20(\delta_e - \delta_{e,c})}_{\dot{h}(\boldsymbol{x}, \boldsymbol{u})} \geq -\alpha \underbrace{(25 - \delta_e)}_{h(\boldsymbol{x})}$$

Example: Safety Filter for F-16



F-16 Pitch Dynamics

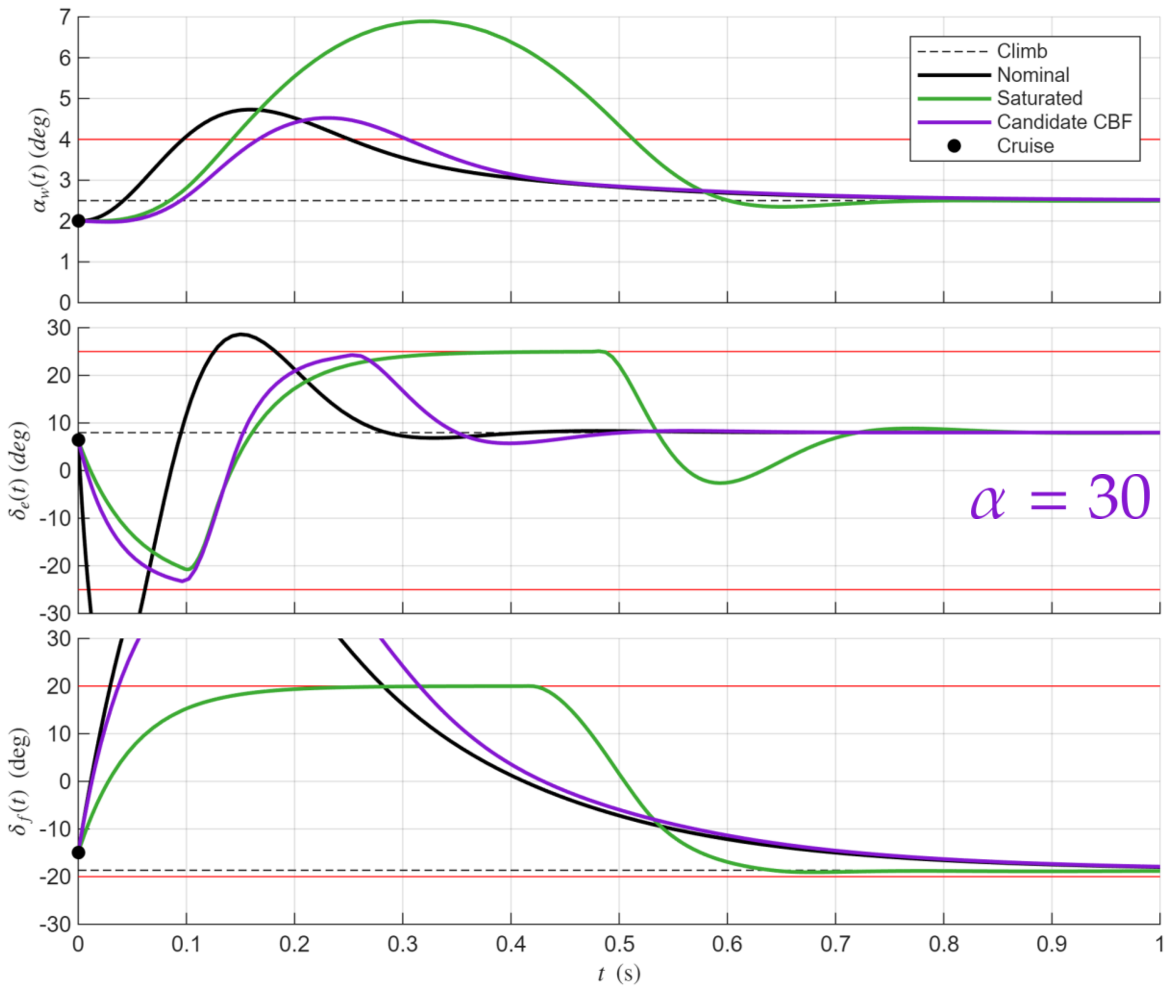
$$x = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

$$u = \begin{bmatrix} \delta_{e,c} & \delta_{f,c} \end{bmatrix}$$

LQR Controller $\kappa(x) = \bar{u}(r) - K(x - \bar{x}(r))$

Saturated LQR $u = \mathrm{sat}\big(\kappa(x)\big)$

Example: Safety Filter for F-16



F-16 Pitch Dynamics

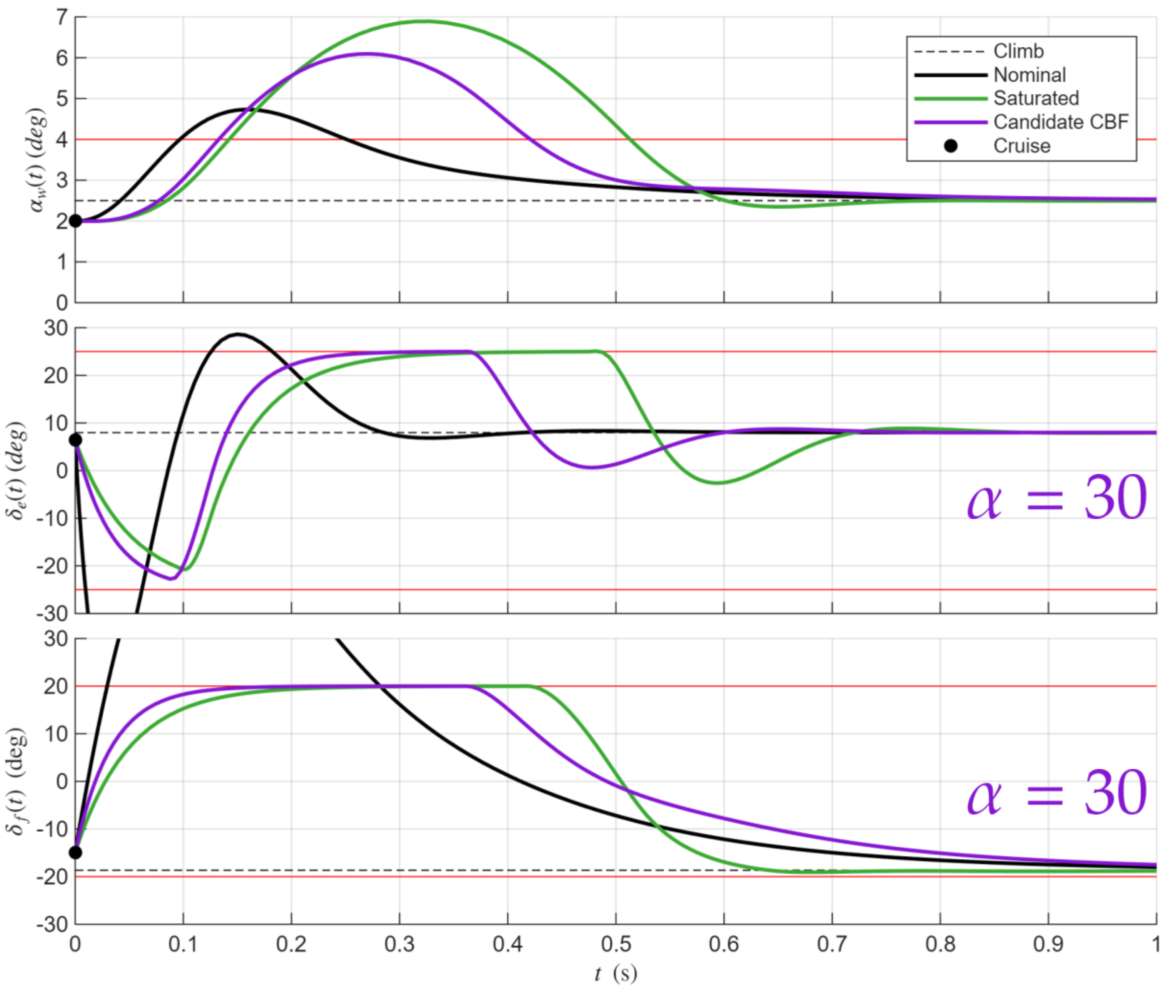
$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

$$\boldsymbol{u} = \begin{bmatrix} \delta_{e,c} & \delta_{f,c} \end{bmatrix}$$

$$\min_{\boldsymbol{u} \in \mathbb{R}^2} \quad \|\boldsymbol{u} - \kappa(\boldsymbol{x})\|^2$$

$$\text{s.t.} \quad 20(\delta_e - \delta_{e,c}) \geq -30(25 - \delta_e)$$

Example: Safety Filter for F-16



F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

$$\boldsymbol{u} = \begin{bmatrix} \delta_{e,c} & \delta_{f,c} \end{bmatrix}$$

$$\begin{aligned} \min_{\boldsymbol{u} \in \mathbb{R}^2} \quad & \|\boldsymbol{u} - \kappa(\boldsymbol{x})\|^2 \\ \text{s.t.} \quad & 20(\delta_e - \delta_{e,c}) \geq -30(25 - \delta_e) \\ & 20(\delta_f - \delta_{f,c}) \geq -30(20 - \delta_f) \end{aligned}$$

Example: Safety Filter for F-16

Naïve CBF Design (alpha)

This procedure yields ***candidate CBFs***.
They provide **no safety guarantees**.

F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

$$\boldsymbol{u} = \begin{bmatrix} \delta_{e,c} & \delta_{f,c} \end{bmatrix}$$

$$4\ \text{deg} \geq \alpha_{\text{w}}$$

Example: Safety Filter for F-16

Naïve CBF Design (alpha)

Stall constraint: $h(\boldsymbol{x}) = 4 - \alpha_w \geq 0$

This procedure yields ***candidate CBFs***.
They provide **no safety guarantees**.

F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$
$$\boldsymbol{u} = \begin{bmatrix} \delta_{e,c} & \delta_{f,c} \end{bmatrix}$$

$4 \text{ deg} \geq \alpha_{\text{w}}$

Example: Safety Filter for F-16

Naïve CBF Design (alpha)

Stall constraint: $h(\boldsymbol{x}) = 4 - \alpha_w \geq 0$

Differentiate: $\frac{\partial h}{\partial x} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \end{bmatrix}$

This procedure yields ***candidate CBFs***.
They provide **no safety guarantees**.

F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

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Example: Safety Filter for F-16

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Differentiate: $\frac{\partial h}{\partial x} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \end{bmatrix}$

Compute: $\dot{h}(\boldsymbol{x}, \boldsymbol{u}) = L_f h(\boldsymbol{x}) + L_g h(\boldsymbol{x})\boldsymbol{u}$

$$L_f h(\boldsymbol{x}) = \frac{\partial h}{\partial \boldsymbol{x}} A\boldsymbol{x}$$

$$L_g h(\boldsymbol{x}) = \frac{\partial h}{\partial \boldsymbol{x}} B$$

This procedure yields ***candidate CBFs***. They provide **no safety guarantees**.

F-16 Pitch Dynamics

$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$

$\boldsymbol{u} = \begin{bmatrix} \delta_{e,c} & \delta_{f,c} \end{bmatrix}$

$f(\boldsymbol{x}) = A\boldsymbol{x} \quad g(\boldsymbol{x}) = B$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -0.87 & 43.22 & -17.25 & -1.58 \\ 0 & 0.99 & -1.34 & -0.17 & -0.25 \\ 0 & 0 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & -20 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 20 & 0 \\ 0 & 20 \end{bmatrix}$$

Example: Safety Filter for F-16

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Compute: $\dot{h}(\boldsymbol{x}, \boldsymbol{u}) = L_f h(\boldsymbol{x}) + L_g h(\boldsymbol{x})\boldsymbol{u}$

$$L_f h(\boldsymbol{x}) = \frac{\partial h}{\partial \boldsymbol{x}} A\boldsymbol{x}$$

$$= -\dot{\theta} + 1.3\alpha_w + 0.2\delta_e + 0.3\delta_f$$

$$L_g h(\boldsymbol{x}) = \frac{\partial h}{\partial \boldsymbol{x}} B$$

This procedure yields ***candidate CBFs***.
They provide **no safety guarantees**.

F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

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Example: Safety Filter for F-16

Naïve CBF Design (alpha)

Stall constraint: $h(\boldsymbol{x}) = 4 - \alpha_w \geq 0$

Differentiate: $\frac{\partial h}{\partial x} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \end{bmatrix}$

Compute: $\dot{h}(\boldsymbol{x}, \boldsymbol{u}) = L_f h(\boldsymbol{x}) + L_g h(\boldsymbol{x})\boldsymbol{u}$

$$L_f h(\boldsymbol{x}) = \frac{\partial h}{\partial \boldsymbol{x}} A\boldsymbol{x}$$
$$= -\dot{\theta} + 1.3\alpha_w + 0.2\delta_e + 0.3\delta_f$$

$$L_g h(\boldsymbol{x}) = \frac{\partial h}{\partial \boldsymbol{x}} B = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

This procedure yields ***candidate CBFs***.
They provide **no safety guarantees**.

F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$
$$\boldsymbol{u} = \begin{bmatrix} \delta_{e,c} & \delta_{f,c} \end{bmatrix}$$
$$f(\boldsymbol{x}) = A\boldsymbol{x} \quad g(\boldsymbol{x}) = B$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -0.87 & 43.22 & -17.25 & -1.58 \\ 0 & 0.99 & -1.34 & -0.17 & -0.25 \\ 0 & 0 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & -20 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 20 & 0 \\ 0 & 20 \end{bmatrix}$$

Example: Safety Filter for F-16

Naïve CBF Design (alpha)

Stall constraint: $h(\boldsymbol{x}) = 4 - \alpha_w \geq 0$

Differentiate: $\frac{\partial h}{\partial x} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \end{bmatrix}$

Compute: $\dot{h}(\boldsymbol{x}, \boldsymbol{u}) = L_f h(\boldsymbol{x}) + L_g h(\boldsymbol{x}) \boldsymbol{u}$

$$L_f h(\boldsymbol{x}) = \frac{\partial h}{\partial \boldsymbol{x}} A \boldsymbol{x}$$

$$= -\dot{\theta} + 1.3\alpha_w + 0.2\delta_e + 0.3\delta_f$$

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F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

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$$f(\boldsymbol{x}) = A\boldsymbol{x} \quad g(\boldsymbol{x}) = B$$

Relative degree > 1

Example: Safety Filter for F-16

Naïve CBF Design (alpha)

Stall constraint: $h(\boldsymbol{x}) = 4 - \alpha_w \geq 0$

Differentiate: $\frac{\partial h}{\partial x} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \end{bmatrix}$

Compute: $\dot{h}(\boldsymbol{x}, \boldsymbol{u}) = L_f h(\boldsymbol{x}) + \cancel{L_g h(\boldsymbol{x})}\boldsymbol{u}$

$$L_f h(\boldsymbol{x}) = \frac{\partial h}{\partial \boldsymbol{x}} A\boldsymbol{x}$$
$$= -\dot{\theta} + 1.3\alpha_w + 0.2\delta_e + 0.3\delta_f$$

$$L_g h(\boldsymbol{x}) = \frac{\partial h}{\partial \boldsymbol{x}} B = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Lost Control Authority

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F-16 Pitch Dynamics

$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$

$\boldsymbol{u} = \begin{bmatrix} \delta_{e,c} & \delta_{f,c} \end{bmatrix}$

$f(\boldsymbol{x}) = A\boldsymbol{x} \quad g(\boldsymbol{x}) = B$

Example: Safety Filter for F-16

Naïve CBF Design (alpha)

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Differentiate: $\frac{\partial h}{\partial x} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \end{bmatrix}$

Compute: $\dot{h}(\boldsymbol{x}, \boldsymbol{u}) = L_f h(\boldsymbol{x})$

Enforce: $L_f h(\boldsymbol{x}) \geq -\alpha\big(h(\boldsymbol{x})\big)$

Lost Control Authority

This procedure yields ***candidate CBFs***.
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F-16 Pitch Dynamics

$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$

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Example: Safety Filter for F-16

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$$\boldsymbol{u} = \begin{bmatrix} \delta_{e,c} & \delta_{f,c} \end{bmatrix}$$

Example: Safety Filter for F-16

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Stall constraint: $h(\boldsymbol{x}) = 4 - \alpha_w \geq 0$

Enforce: $L_f h(\boldsymbol{x}) \geq -\alpha\big(h(\boldsymbol{x})\big)$

Treat as new constraint…

Define: $h_1(\boldsymbol{x}) = L_f h(\boldsymbol{x}) + \alpha\big(h(\boldsymbol{x})\big) \geq 0$

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Example: Safety Filter for F-16

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Treat as new constraint…

Define: $h_1(\boldsymbol{x}) = L_f h(\boldsymbol{x}) + \alpha\big(h(\boldsymbol{x})\big) \geq 0$

Differentiate: $\frac{\partial h_1}{\partial x} = \frac{\partial h}{\partial x}(A + \alpha I_n)$

⚠ This procedure yields ***candidate CBFs***. They provide **no safety guarantees**.

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$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

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$$\frac{\partial h}{\partial x} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

Example: Safety Filter for F-16

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Differentiate: $\frac{\partial h_1}{\partial x} = \frac{\partial h}{\partial x}(A + \alpha I_n)$

Compute: $\dot{h}_1(\boldsymbol{x}, \boldsymbol{u}) = L_f h_1(\boldsymbol{x}) + L_g h_1(\boldsymbol{x})\boldsymbol{u}$

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Example: Safety Filter for F-16

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Differentiate: $\frac{\partial h_1}{\partial x} = \frac{\partial h}{\partial x}(A + \alpha I_n)$

Compute: $\dot{h}_1(\boldsymbol{x},\boldsymbol{u}) = L_f h_1(\boldsymbol{x}) + L_g h_1(\boldsymbol{x})\boldsymbol{u}$

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Example: Safety Filter for F-16

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Stall constraint: $h(\boldsymbol{x}) = 4 - \alpha_w \geq 0$

Enforce: $L_f h(\boldsymbol{x}) \geq -\alpha\big(h(\boldsymbol{x})\big)$

Treat as new constraint…

Define: $h_1(\boldsymbol{x}) = L_f h(\boldsymbol{x}) + \alpha\big(h(\boldsymbol{x})\big) \geq 0$

Differentiate: $\frac{\partial h_1}{\partial x} = \boxed{\frac{\partial h}{\partial x}(A + \alpha I_n)}$

Compute: $\dot{h}_1(\boldsymbol{x},\boldsymbol{u}) = L_f h_1(\boldsymbol{x}) + L_g h_1(\boldsymbol{x})\boldsymbol{u}$

$$L_g h_1(\boldsymbol{x}) = \frac{\partial h_1}{\partial \boldsymbol{x}} B = \boxed{\frac{\partial h}{\partial \boldsymbol{x}}(A + \alpha I_n)} B$$

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$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

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$$\frac{\partial h}{\partial x} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

Example: Safety Filter for F-16

Naïve CBF Design (alpha)

Stall constraint: $h(\boldsymbol{x}) = 4 - \alpha_w \geq 0$

Enforce: $L_f h(\boldsymbol{x}) \geq -\alpha\big(h(\boldsymbol{x})\big)$

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Differentiate: $\frac{\partial h_1}{\partial x} = \frac{\partial h}{\partial x}(A + \alpha I_n)$

Compute: $\dot{h}_1(\boldsymbol{x}, \boldsymbol{u}) = L_f h_1(\boldsymbol{x}) + L_g h_1(\boldsymbol{x})\boldsymbol{u}$

$$L_g h_1(\boldsymbol{x}) = \frac{\partial h_1}{\partial \boldsymbol{x}} B = \frac{\partial h}{\partial \boldsymbol{x}} AB + \alpha \frac{\partial h}{\partial \boldsymbol{x}} B$$

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$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

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Example: Safety Filter for F-16

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Define: $h_1(\boldsymbol{x}) = L_f h(\boldsymbol{x}) + \alpha\big(h(\boldsymbol{x})\big) \geq 0$

Differentiate: $\frac{\partial h_1}{\partial x} = \frac{\partial h}{\partial x}(A + \alpha I_n)$

Compute: $\dot{h}_1(\boldsymbol{x}, \boldsymbol{u}) = L_f h_1(\boldsymbol{x}) + L_g h_1(\boldsymbol{x})\boldsymbol{u}$

$$L_g h_1(\boldsymbol{x}) = \frac{\partial h_1}{\partial \boldsymbol{x}} B = \frac{\partial h}{\partial \boldsymbol{x}} AB + \cancel{\alpha \frac{\partial h}{\partial \boldsymbol{x}} B}$$

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$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

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$$\frac{\partial h}{\partial x} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

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Example: Safety Filter for F-16

Naïve CBF Design (alpha)

Stall constraint: $h(\boldsymbol{x}) = 4 - \alpha_w \geq 0$

Enforce: $L_f h(\boldsymbol{x}) \geq -\alpha\big(h(\boldsymbol{x})\big)$

Treat as new constraint…

Define: $h_1(\boldsymbol{x}) = L_f h(\boldsymbol{x}) + \alpha\big(h(\boldsymbol{x})\big) \geq 0$

Differentiate: $\frac{\partial h_1}{\partial x} = \frac{\partial h}{\partial x}(A + \alpha I_n)$

Compute: $\dot{h}_1(\boldsymbol{x}, \boldsymbol{u}) = L_f h_1(\boldsymbol{x}) + L_g h_1(\boldsymbol{x})\boldsymbol{u}$

$$L_g h_1(\boldsymbol{x}) = \frac{\partial h_1}{\partial \boldsymbol{x}} B = \begin{bmatrix} 3.4 & 5.0 \end{bmatrix}$$

This procedure yields ***candidate CBFs***.
They provide **no safety guarantees**.

F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$
$$\boldsymbol{u} = \begin{bmatrix} \delta_{e,c} & \delta_{f,c} \end{bmatrix}$$

Recovered Control Authority

Example: Safety Filter for F-16

Naïve CBF Design (alpha)

Stall constraint: $h(\boldsymbol{x}) = 4 - \alpha_w \geq 0$

Enforce: $L_f h(\boldsymbol{x}) \geq -\alpha\big(h(\boldsymbol{x})\big)$

Treat as new constraint...

Define: $h_1(\boldsymbol{x}) = L_f h(\boldsymbol{x}) + \alpha\big(h(\boldsymbol{x})\big) \geq 0$

Differentiate: $\frac{\partial h_1}{\partial x} = \frac{\partial h}{\partial x}(A + \alpha I_n)$

Compute: $\dot{h}_1(\boldsymbol{x}, \boldsymbol{u}) = L_f h_1(\boldsymbol{x}) + L_g h_1(\boldsymbol{x})\boldsymbol{u}$

Enforce: $\dot{h}_1(\boldsymbol{x}, \boldsymbol{u}) \geq -\alpha_1\big(h_1(\boldsymbol{x})\big)$

This procedure yields ***candidate CBFs***.
They provide **no safety guarantees**.

F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

$$\boldsymbol{u} = \begin{bmatrix} \delta_{e,c} & \delta_{f,c} \end{bmatrix}$$

$$\min_{\boldsymbol{u} \in \mathbb{R}^2} \quad \|\boldsymbol{u} - \kappa(\boldsymbol{x})\|^2$$
$$\text{s.t.} \quad \dot{h}_1(\boldsymbol{x}, \boldsymbol{u}) \geq -\alpha_1\big(h_1(\boldsymbol{x})\big)$$

Example: Safety Filter for F-16

Naïve CBF Design (alpha)

Stall constraint: $h(\boldsymbol{x}) = 4 - \alpha_w \geq 0$

Enforce: $L_f h(\boldsymbol{x}) \geq -\alpha\big(h(\boldsymbol{x})\big)$

Treat as new constraint…

Define: $h_1(\boldsymbol{x}) = L_f h(\boldsymbol{x}) + \alpha\big(h(\boldsymbol{x})\big) \geq 0$

Differentiate: $\frac{\partial h_1}{\partial x} = \frac{\partial h}{\partial x}(A + \alpha I_n)$

Compute: $\dot{h}_1(\boldsymbol{x}, \boldsymbol{u}) = L_f h_1(\boldsymbol{x}) + L_g h_1(\boldsymbol{x})\boldsymbol{u}$

Enforce: $\dot{h}_1(\boldsymbol{x}, \boldsymbol{u}) \geq -\alpha_1\big(h_1(\boldsymbol{x})\big)$

This procedure yields ***candidate CBFs***.
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F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

$$\boldsymbol{u} = \begin{bmatrix} \delta_{e,c} & \delta_{f,c} \end{bmatrix}$$

$$\min_{\boldsymbol{u} \in \mathbb{R}^2} \quad \|\boldsymbol{u} - \kappa(\boldsymbol{x})\|^2$$
$$\text{s.t.} \quad A_{QP}\boldsymbol{u} \leq b_{QP}(\boldsymbol{x})$$

Example: Safety Filter for F-16

Naïve CBF Design (alpha)

Stall constraint: $h(\boldsymbol{x}) = 4 - \alpha_w \geq 0$

Enforce: $L_f h(\boldsymbol{x}) \geq -\alpha\big(h(\boldsymbol{x})\big)$

Treat as new constraint…

Define: $h_1(\boldsymbol{x}) = L_f h(\boldsymbol{x}) + \alpha\big(h(\boldsymbol{x})\big) \geq 0$

Differentiate: $\frac{\partial h_1}{\partial x} = \frac{\partial h}{\partial x}(A + \alpha I_n)$

Compute: $\dot{h}_1(\boldsymbol{x},\boldsymbol{u}) = L_f h_1(\boldsymbol{x}) + L_g h_1(\boldsymbol{x})\boldsymbol{u}$

Enforce: $\dot{h}_1(\boldsymbol{x},\boldsymbol{u}) \geq -\alpha_1\big(h_1(\boldsymbol{x})\big)$

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$$\boldsymbol{x} = \begin{bmatrix} \theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f \end{bmatrix}$$

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$$\min_{\boldsymbol{u} \in \mathbb{R}^2} \quad \|\boldsymbol{u} - \kappa(\boldsymbol{x})\|^2$$
$$\text{s.t.} \quad A_{QP}\boldsymbol{u} \leq b_{QP}(\boldsymbol{x})$$

$$A_{QP} = -L_g h_1(\boldsymbol{x}) = -\begin{bmatrix} 3.4 & 5.0 \end{bmatrix}$$

Example: Safety Filter for F-16

Naïve CBF Design (alpha)

Stall constraint: $h(\boldsymbol{x}) = 4 - \alpha_w \geq 0$

Enforce: $L_f h(\boldsymbol{x}) \geq -\alpha\big(h(\boldsymbol{x})\big)$

Treat as new constraint…

Define: $h_1(\boldsymbol{x}) = L_f h(\boldsymbol{x}) + \alpha\big(h(\boldsymbol{x})\big) \geq 0$

Differentiate: $\frac{\partial h_1}{\partial x} = \frac{\partial h}{\partial x}(A + \alpha I_n)$

Compute: $\dot{h}_1(\boldsymbol{x},\boldsymbol{u}) = L_f h_1(\boldsymbol{x}) + L_g h_1(\boldsymbol{x})\boldsymbol{u}$

Enforce: $\dot{h}_1(\boldsymbol{x},\boldsymbol{u}) \geq -\alpha_1\big(h_1(\boldsymbol{x})\big)$

This procedure yields ***candidate CBFs***. They provide **no safety guarantees**.

F-16 Pitch Dynamics

$$\boldsymbol{x} = \begin{bmatrix}\theta & \dot{\theta} & \alpha_w & \delta_e & \delta_f\end{bmatrix}$$

$$\boldsymbol{u} = \begin{bmatrix}\delta_{e,c} & \delta_{f,c}\end{bmatrix}$$

$$\begin{aligned} \min_{\boldsymbol{u}\in\mathbb{R}^2} \quad & \|\boldsymbol{u} - \kappa(\boldsymbol{x})\|^2 \\ \text{s.t.} \quad & A_{QP}\boldsymbol{u} \leq b_{QP}(\boldsymbol{x}) \end{aligned}$$

$$A_{QP} = -L_g h_1(\boldsymbol{x}) = -\begin{bmatrix}3.4 & 5.0\end{bmatrix}$$

$$b_{QP}(\boldsymbol{x}) = (\alpha + \alpha_1)\frac{\partial h}{\partial x}A\boldsymbol{x} + \alpha\alpha_1 h(\boldsymbol{x}) + \frac{\partial h}{\partial x}A^2\boldsymbol{x}$$

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Example: Safety Filter for F-16

Naïve CBF Design (alpha)

Stall constraint: $h(\boldsymbol{x}) = 4 - \alpha_w \geq 0$

Enforce: $L_f h(\boldsymbol{x}) \geq -\alpha\big(h(\boldsymbol{x})\big)$

Treat as new constraint…

Define: $h_1(\boldsymbol{x}) = L_f h(\boldsymbol{x}) + \alpha\big(h(\boldsymbol{x})\big) \geq 0$

Differentiate: $\frac{\partial h_1}{\partial x} = \frac{\partial h}{\partial x}(A + \alpha I_n)$

Compute: $\dot{h}_1(\boldsymbol{x}, \boldsymbol{u}) = L_f h_1(\boldsymbol{x}) + L_g h_1(\boldsymbol{x})\boldsymbol{u}$

Enforce: $\dot{h}_1(\boldsymbol{x}, \boldsymbol{u}) \geq -\alpha_1\big(h_1(\boldsymbol{x})\big)$

Tune: $\alpha, \alpha_1 > 0$

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$$\boldsymbol{u} = \begin{bmatrix}\delta_{e,c} & \delta_{f,c}\end{bmatrix}$$

Roots of polynomial, eigenvalues (2x2)

$$\min_{\boldsymbol{u}\in\mathbb{R}^2} \quad \|\boldsymbol{u} - \kappa(\boldsymbol{x})\|^2$$
$$\text{s.t.} \quad A_{QP}\boldsymbol{u} \leq b_{QP}(\boldsymbol{x})$$

$$A_{QP} = -L_g h_1(\boldsymbol{x}) = -\begin{bmatrix}3.4 & 5.0\end{bmatrix}$$

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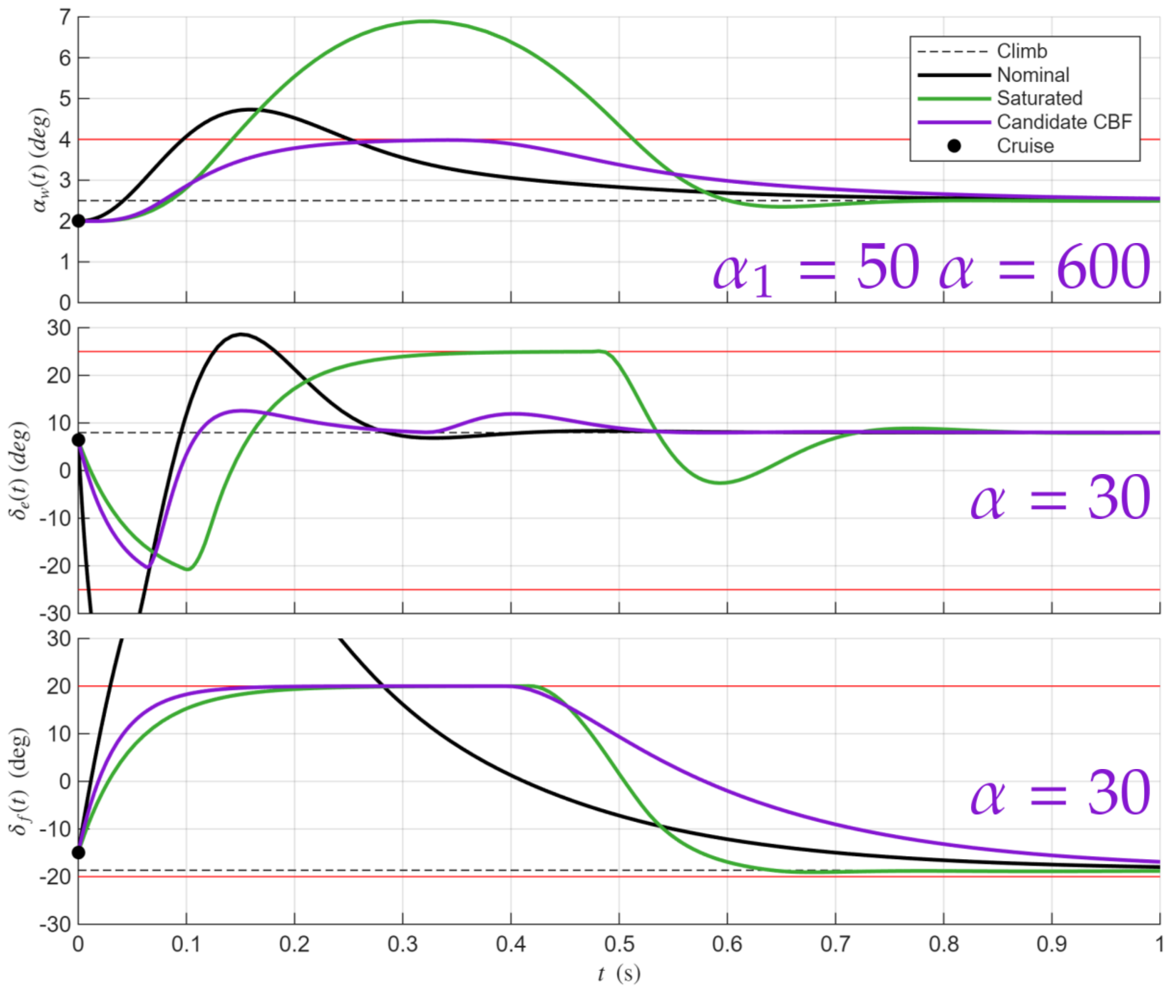
Example: Safety Filter for F-16



F-16 Pitch Dynamics

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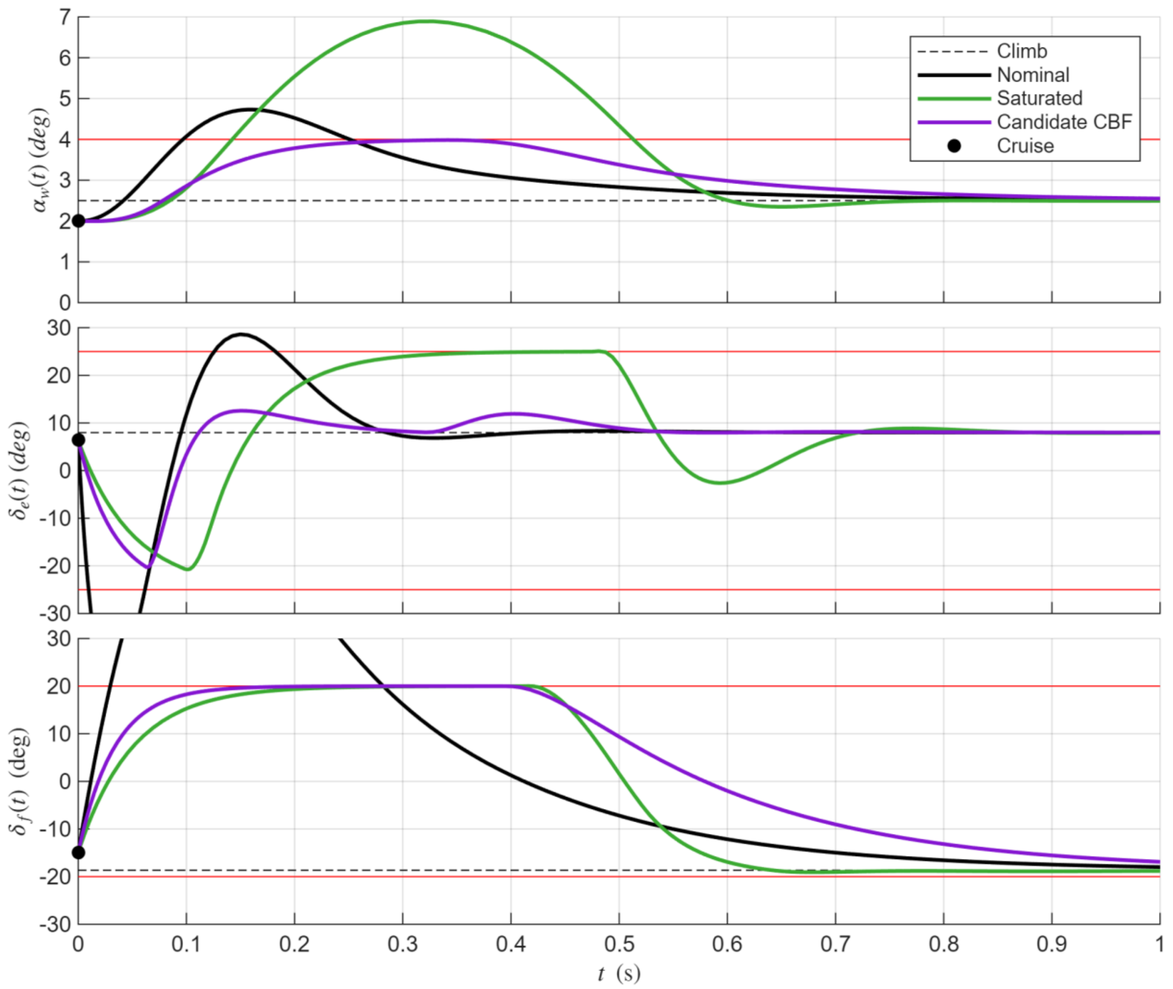
$$\boldsymbol{u} = \begin{bmatrix} \delta_{e,c} & \delta_{f,c} \end{bmatrix}$$

$$\begin{aligned} \min_{\boldsymbol{u} \in \mathbb{R}^2} \quad & \|\boldsymbol{u} - \kappa(\boldsymbol{x})\|^2 \\ \text{s.t.} \quad & 20(\delta_e - \delta_{e,c}) \geq -30(25 - \delta_e) \\ & 20(\delta_f - \delta_{f,c}) \geq -30(20 - \delta_f) \\ & \dot{h}_1(\boldsymbol{x}, \boldsymbol{u}) \geq -\alpha_1\big(h_1(\boldsymbol{x})\big) \end{aligned}$$

mosek

3.5 ms solvetime

Example: Safety Filter for F-16



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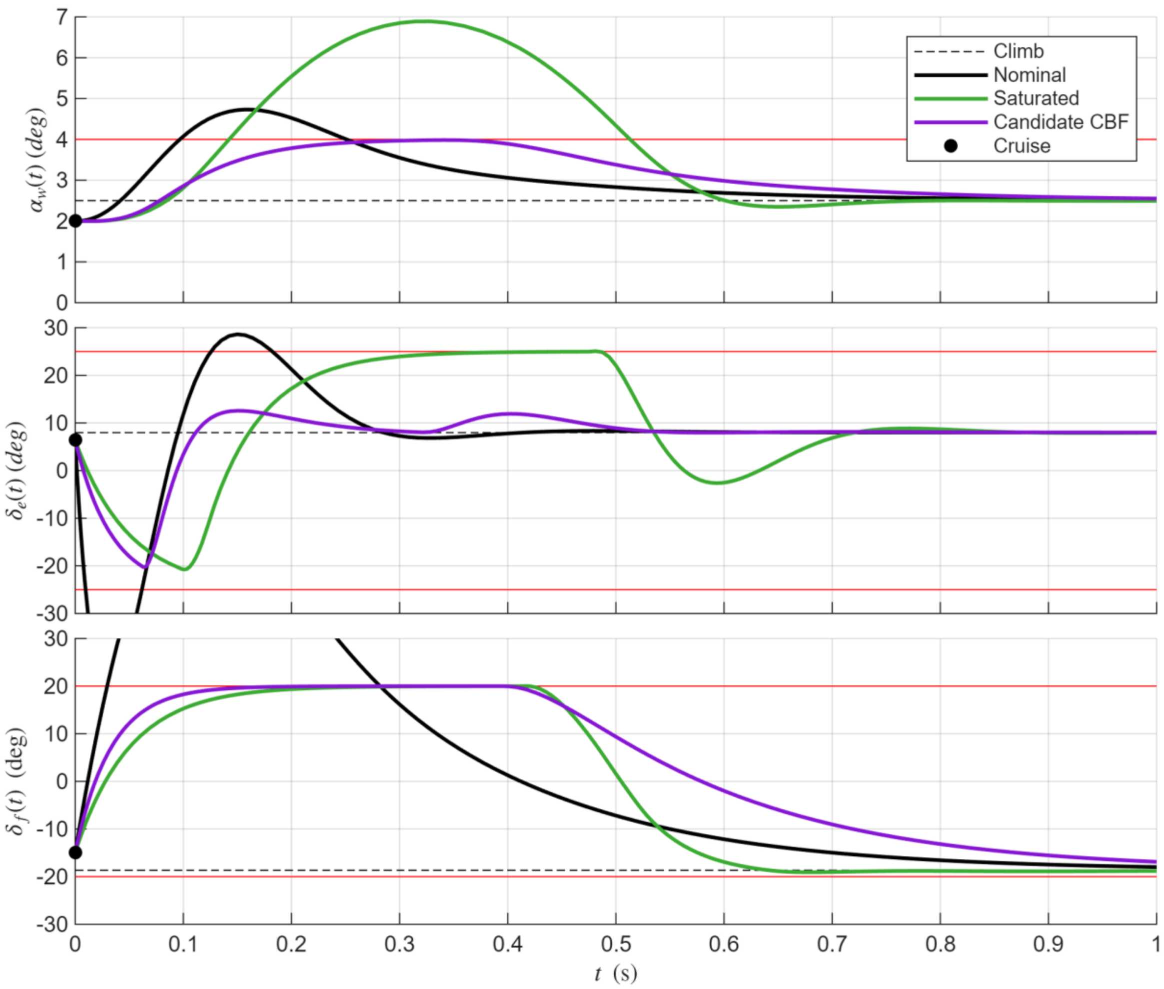
Example: Safety Filter for F-16



F-16 Pitch Dynamics

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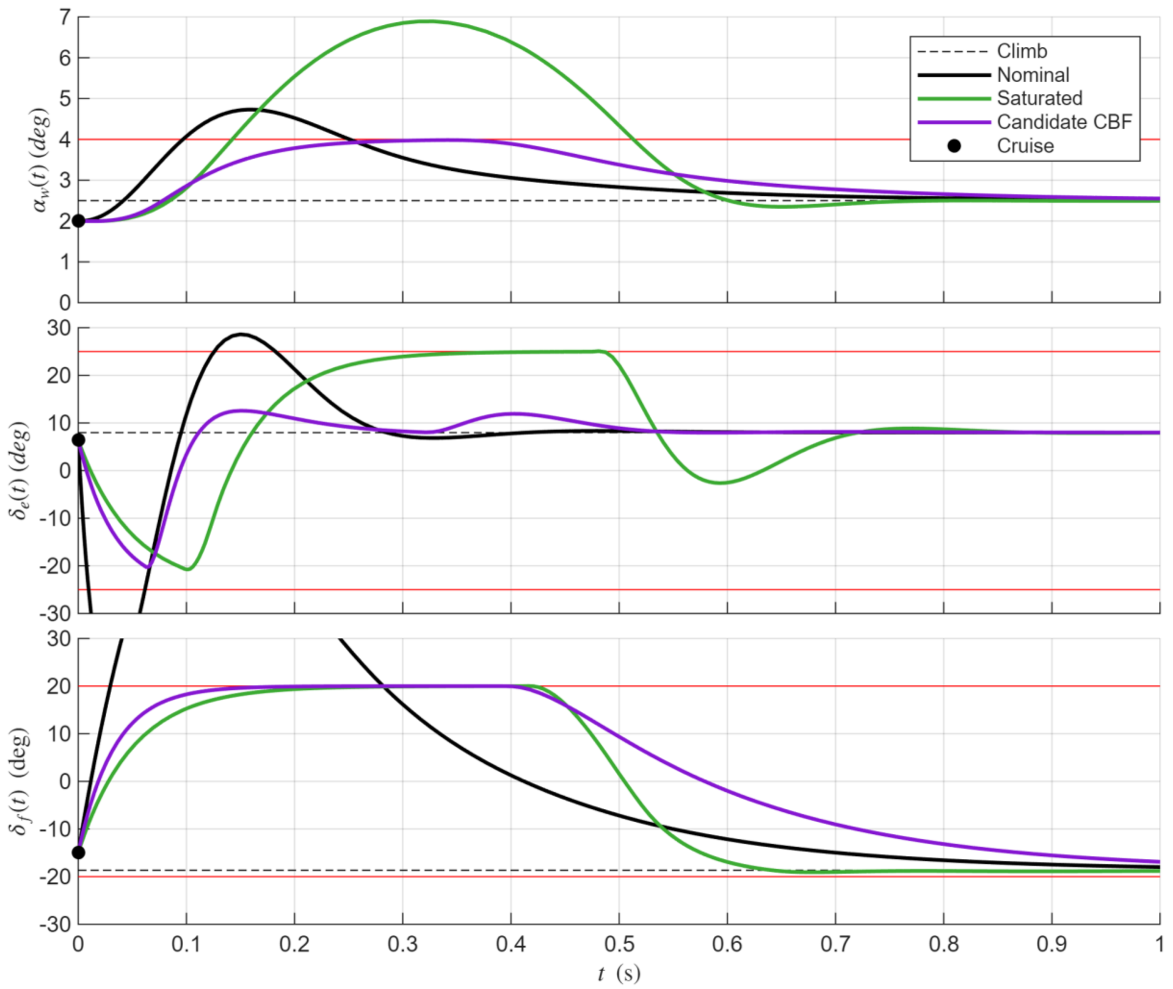
This procedure yields ***candidate CBFs***. They provide **no safety guarantees**.

Example: Safety Filter for F-16



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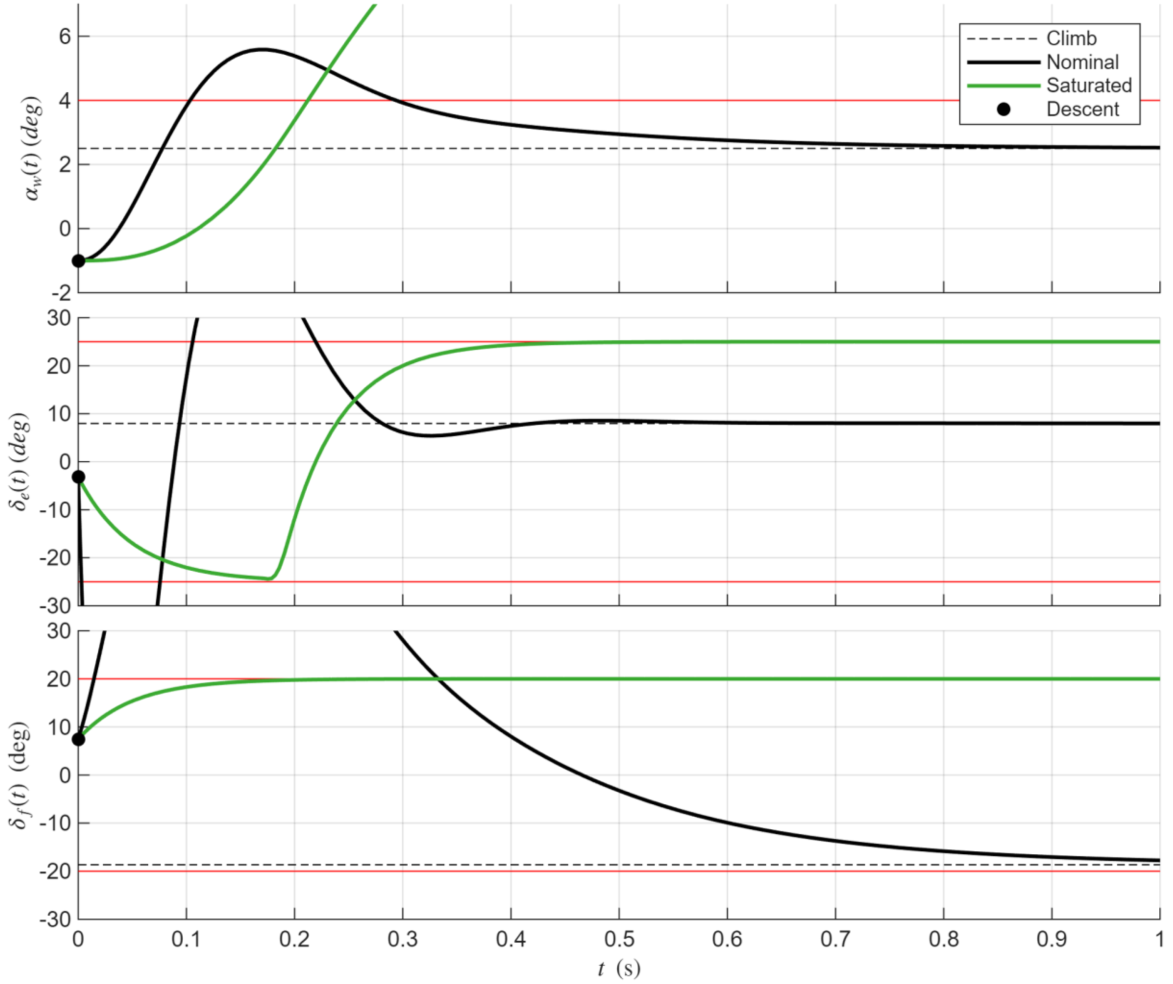

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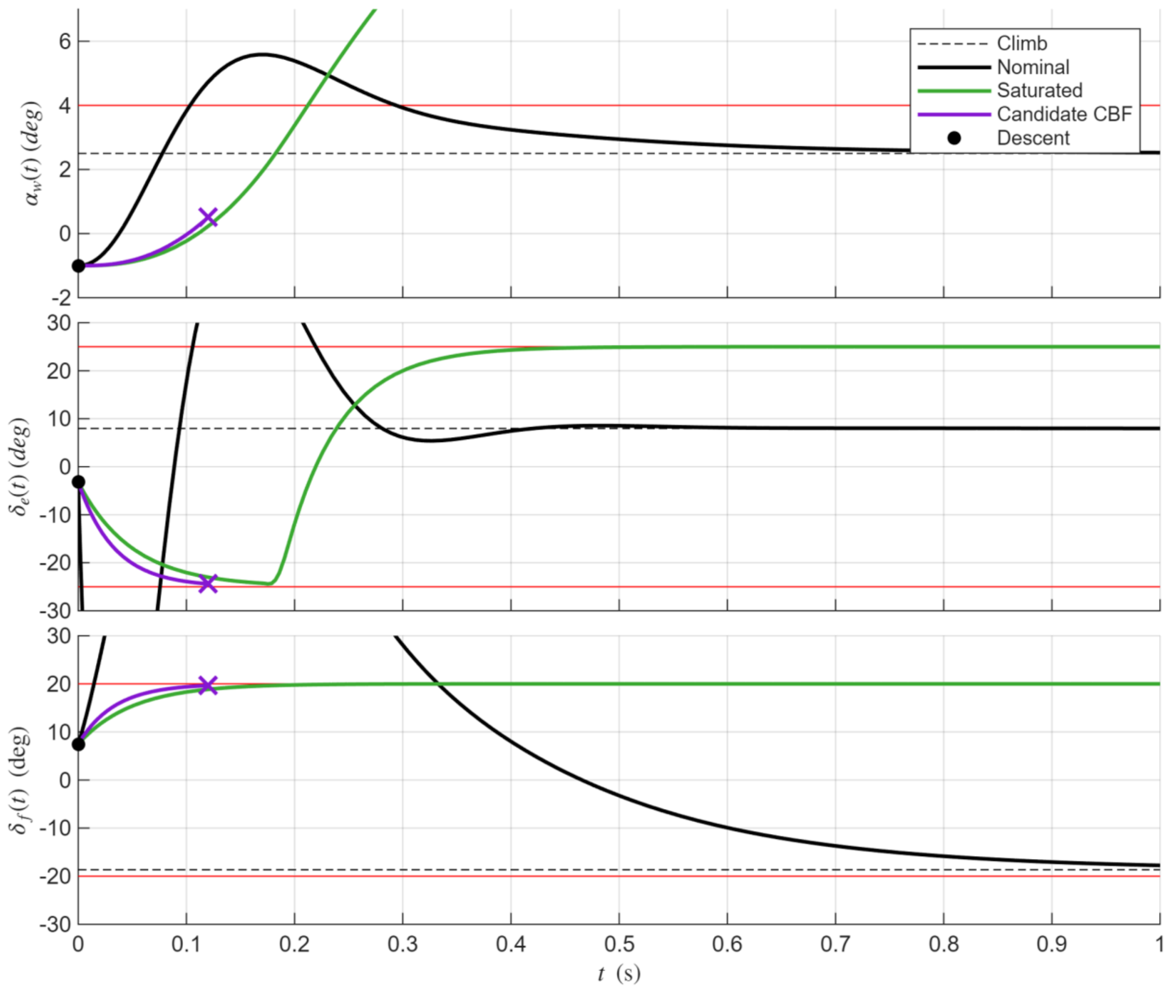

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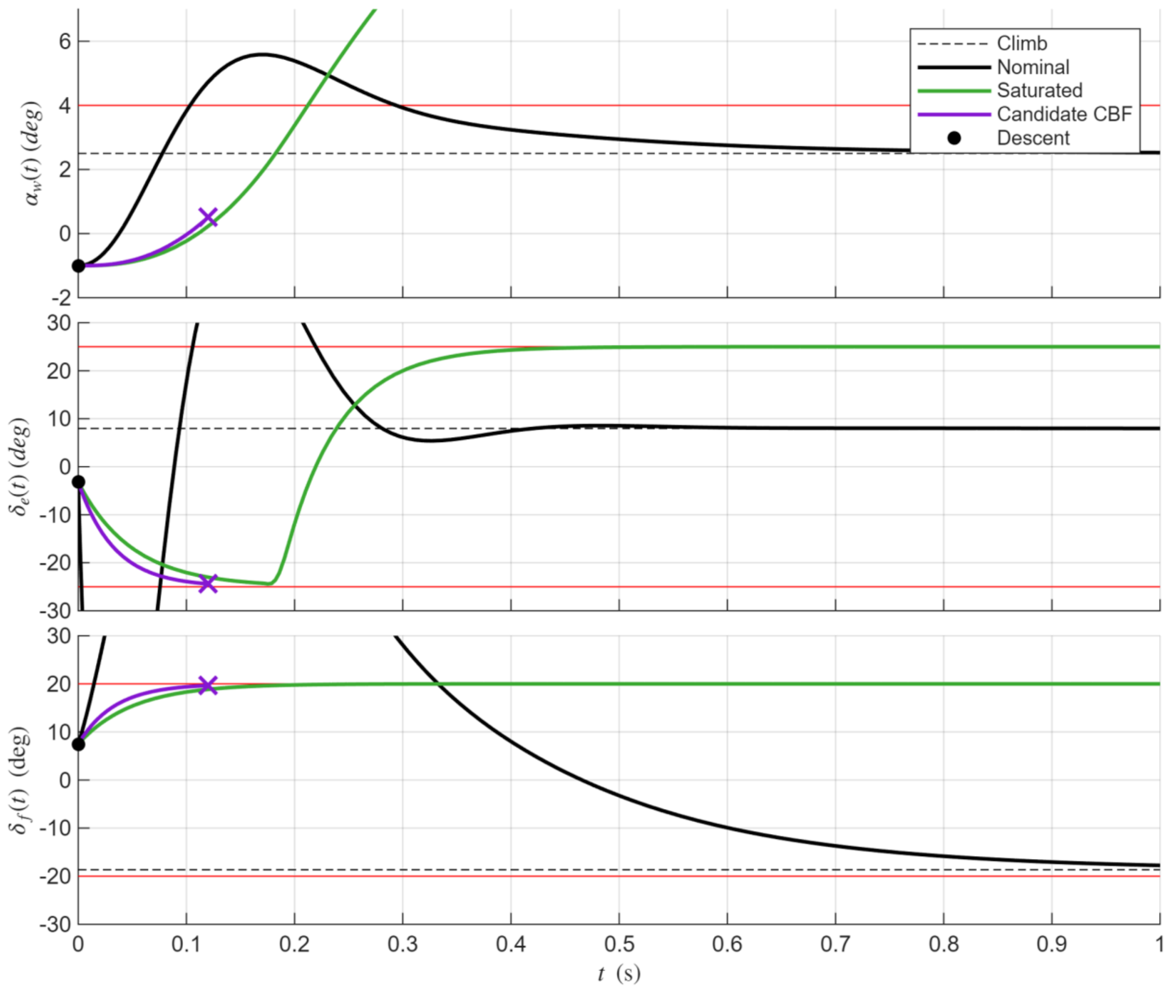

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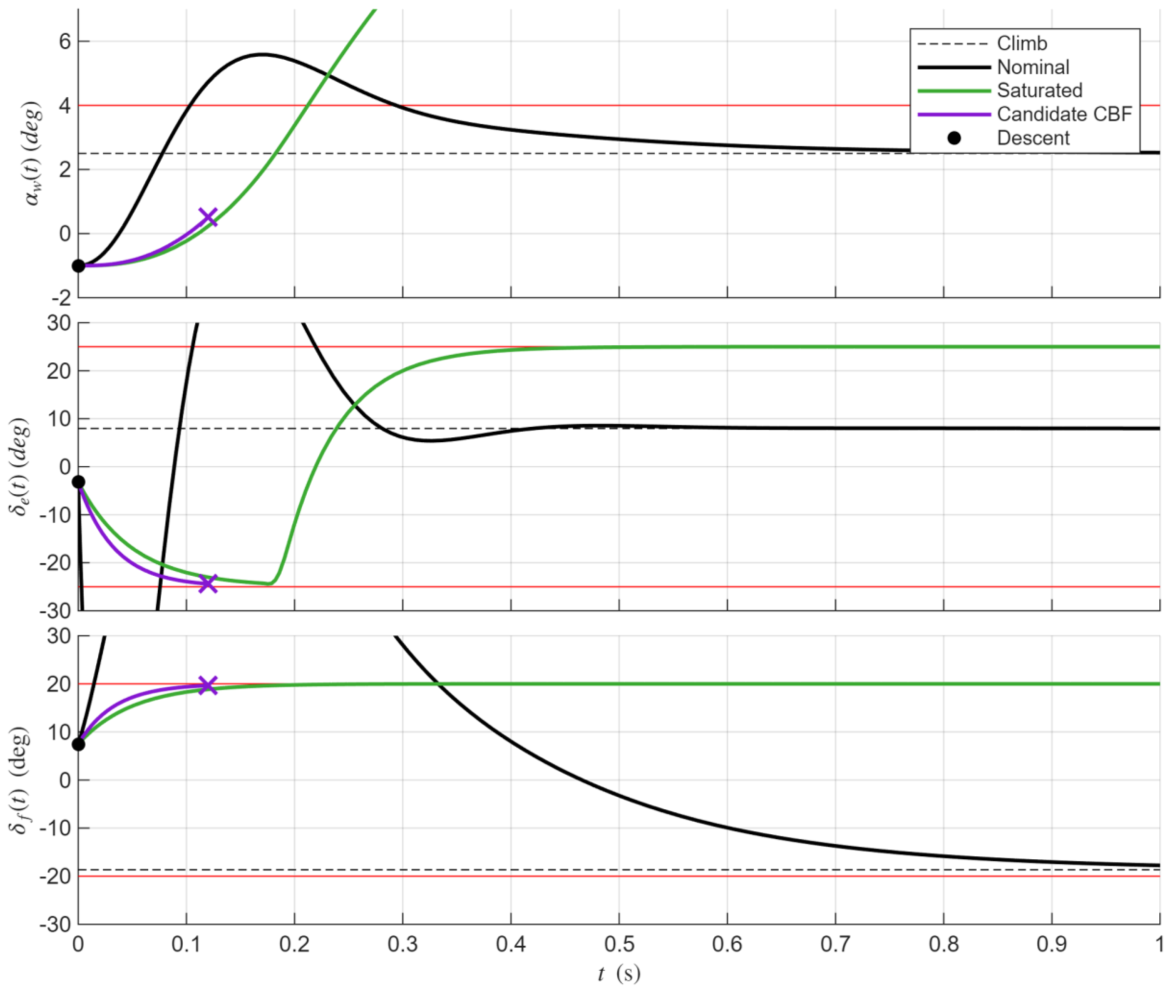

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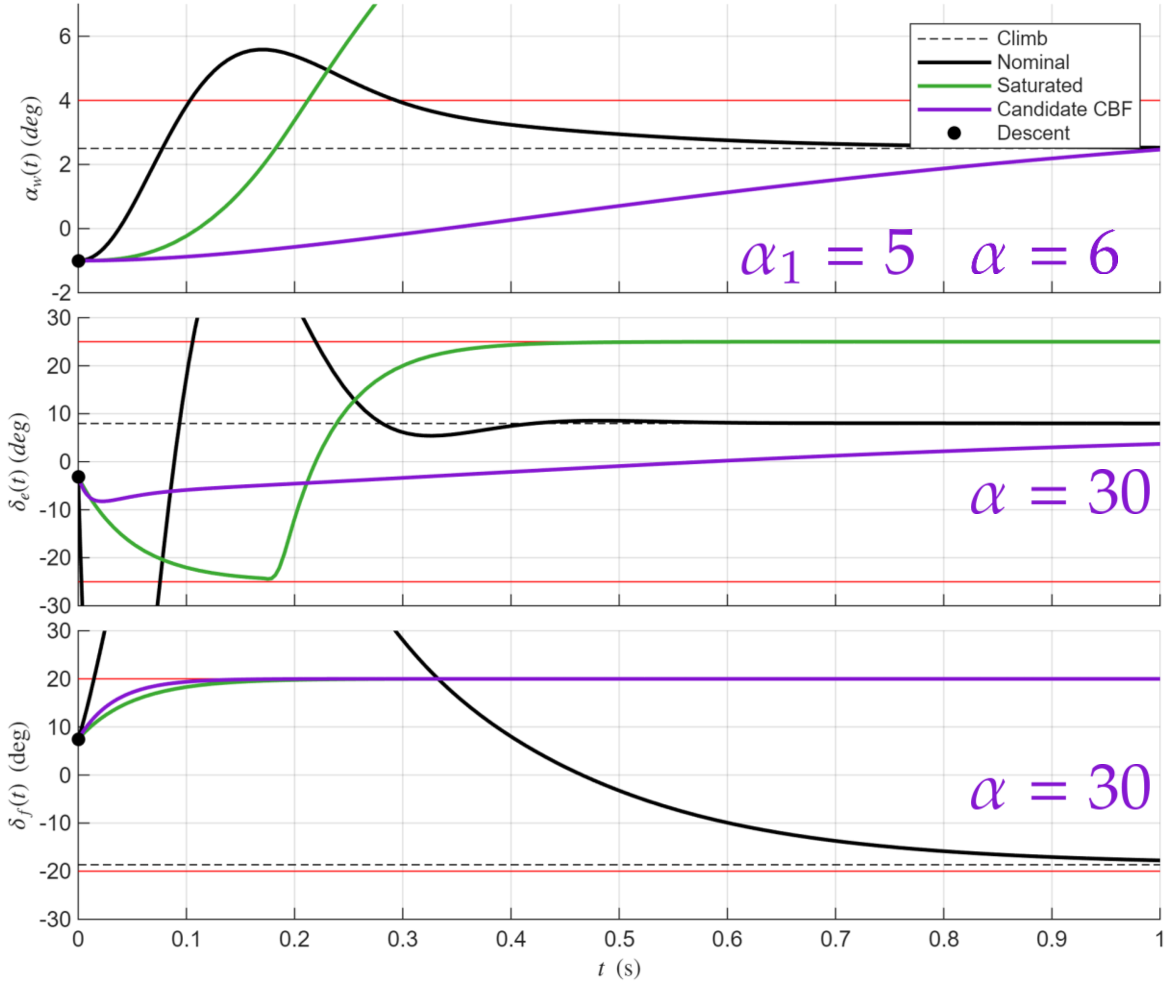

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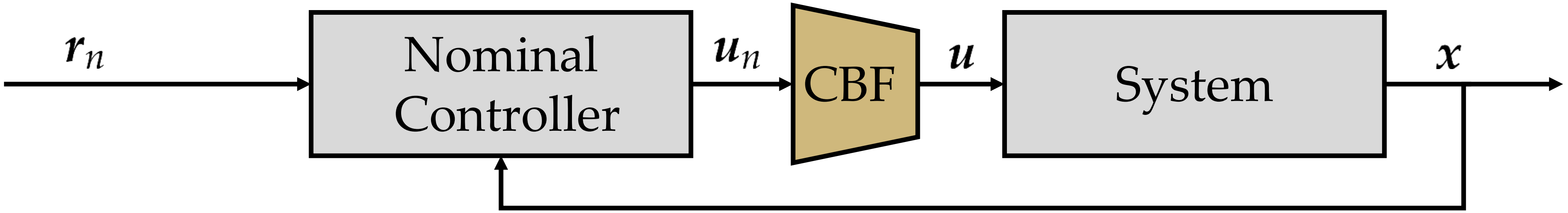

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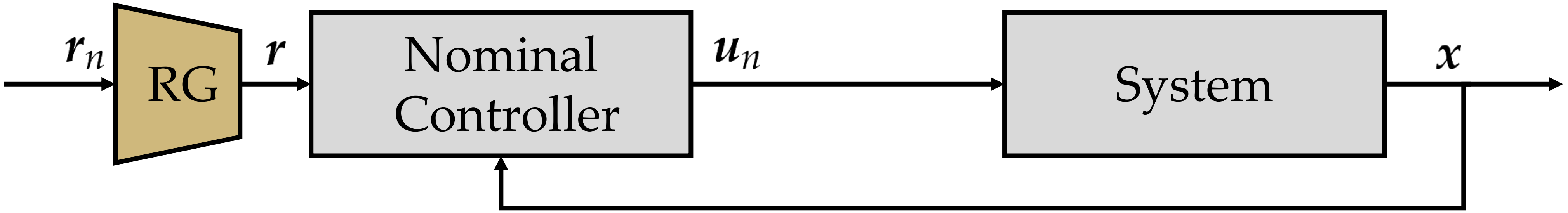
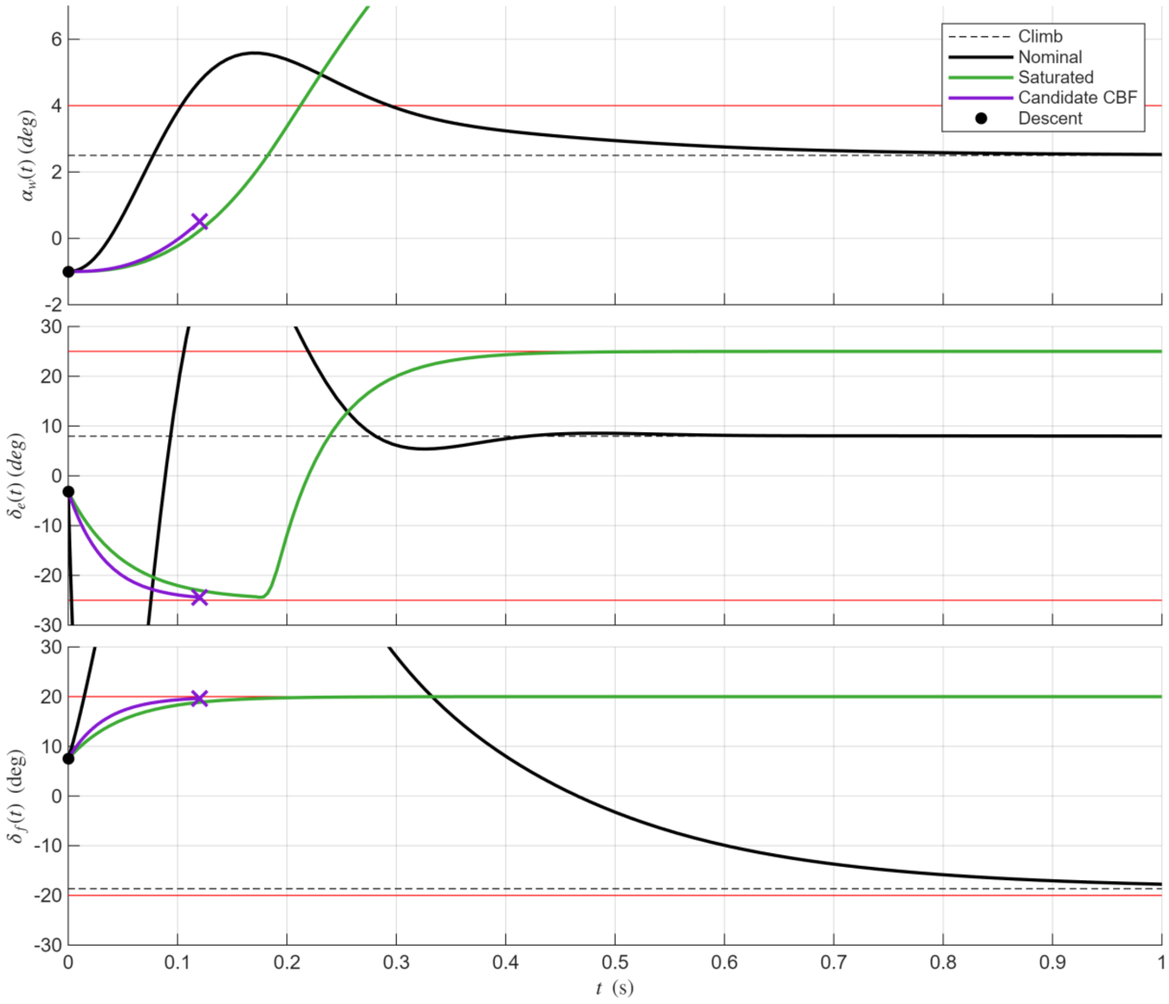


My Work: DSMs are CBFs



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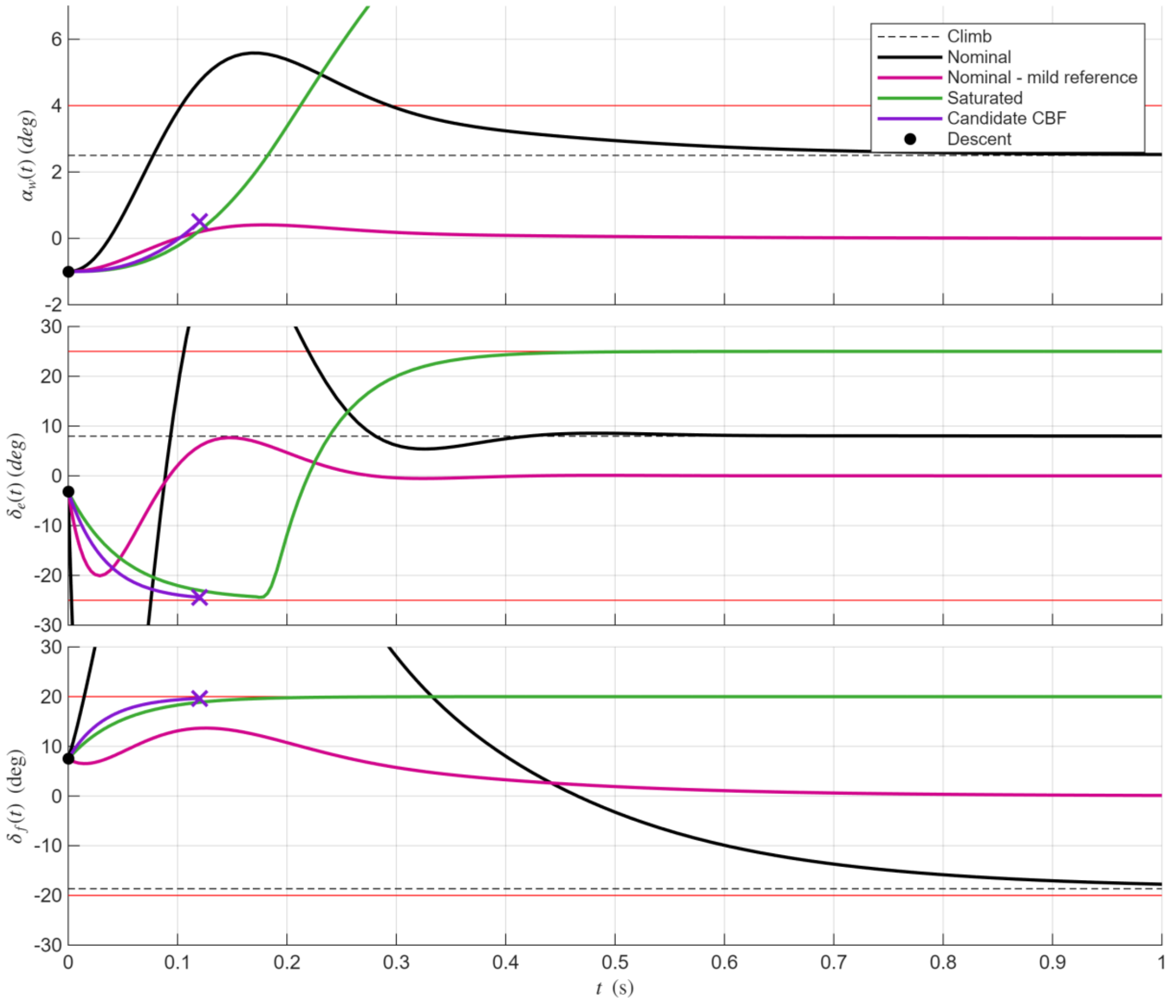

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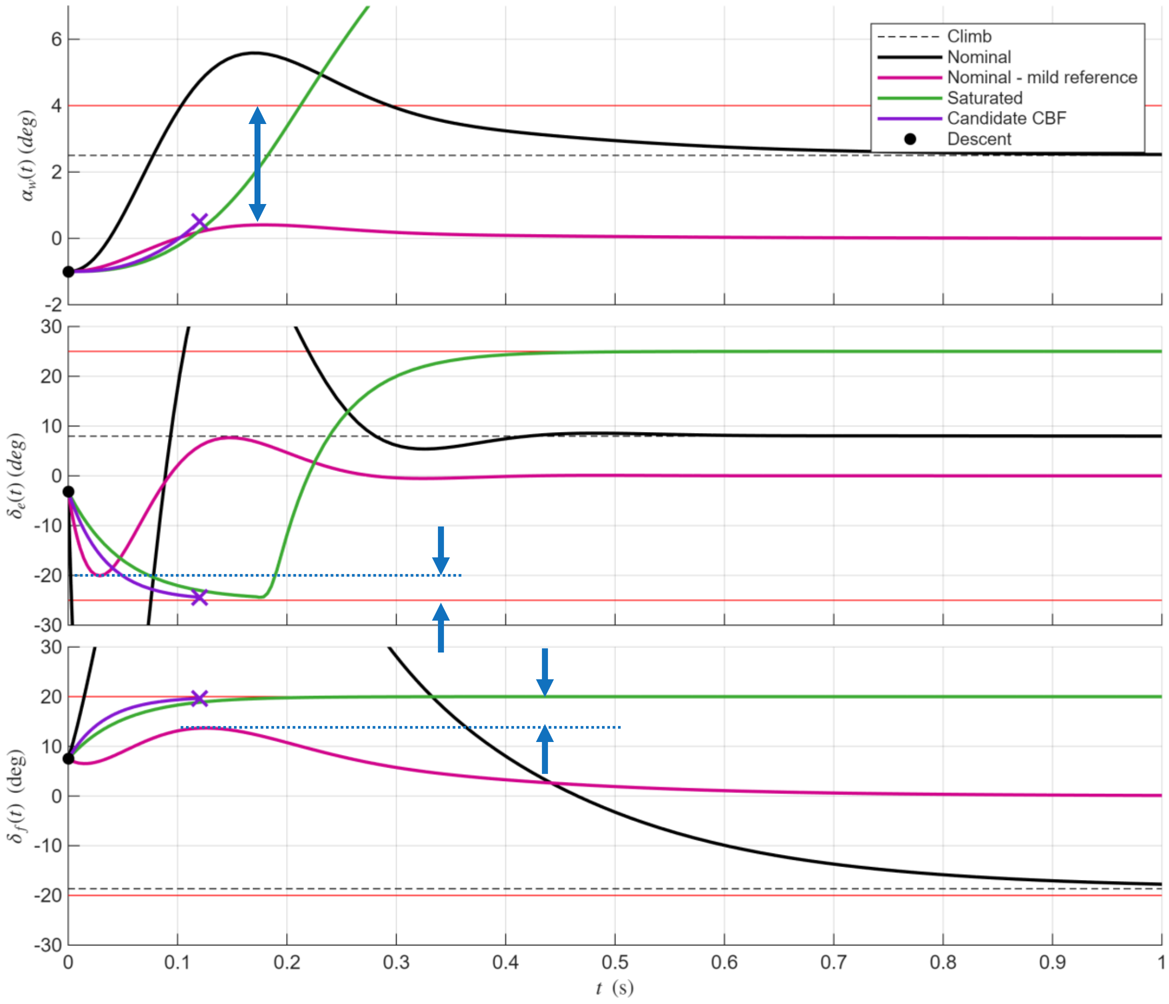

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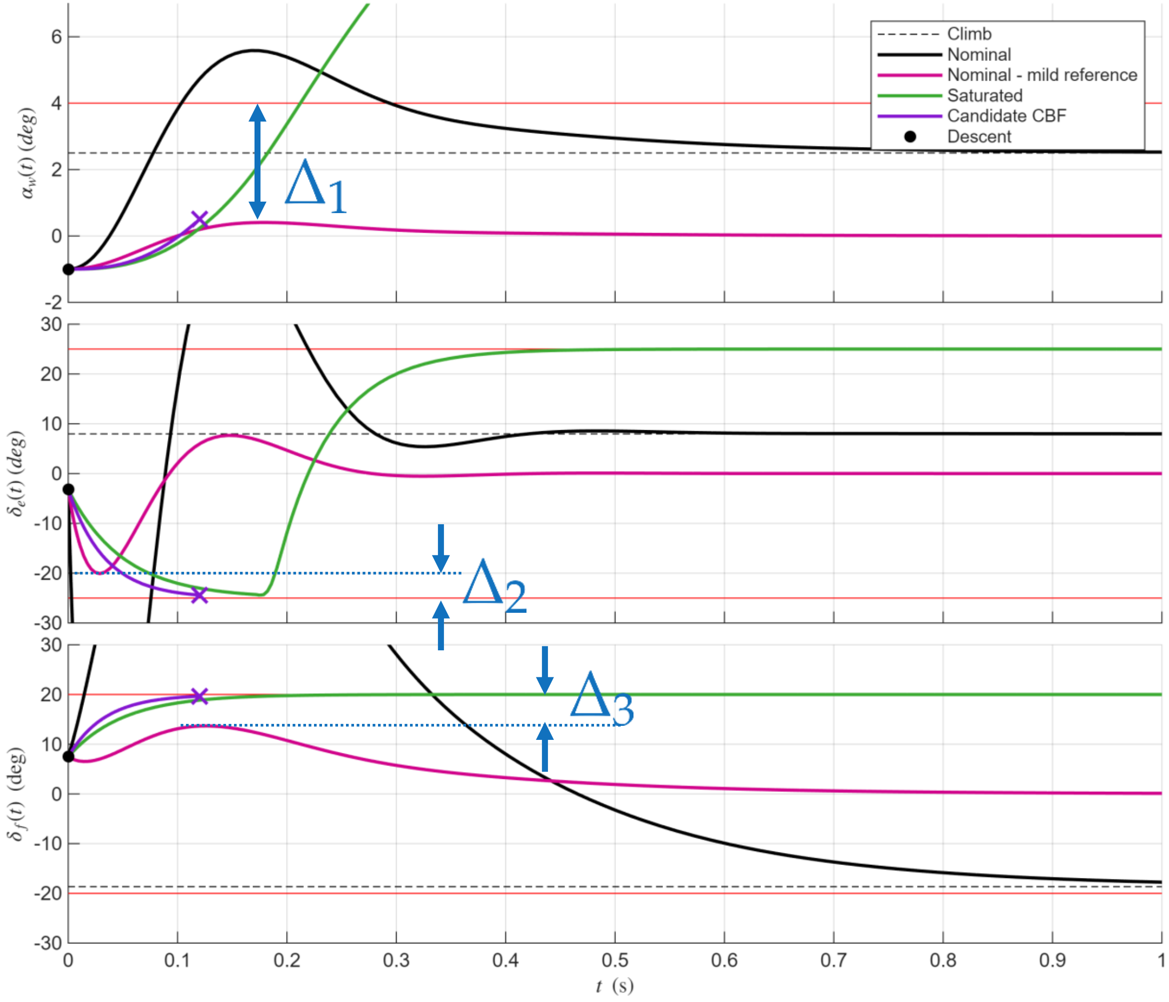

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$$\Delta(\boldsymbol{x}, v) = \inf_{\tau \in [0,\infty)} h\big(\Phi(\tau, v)\big)$$

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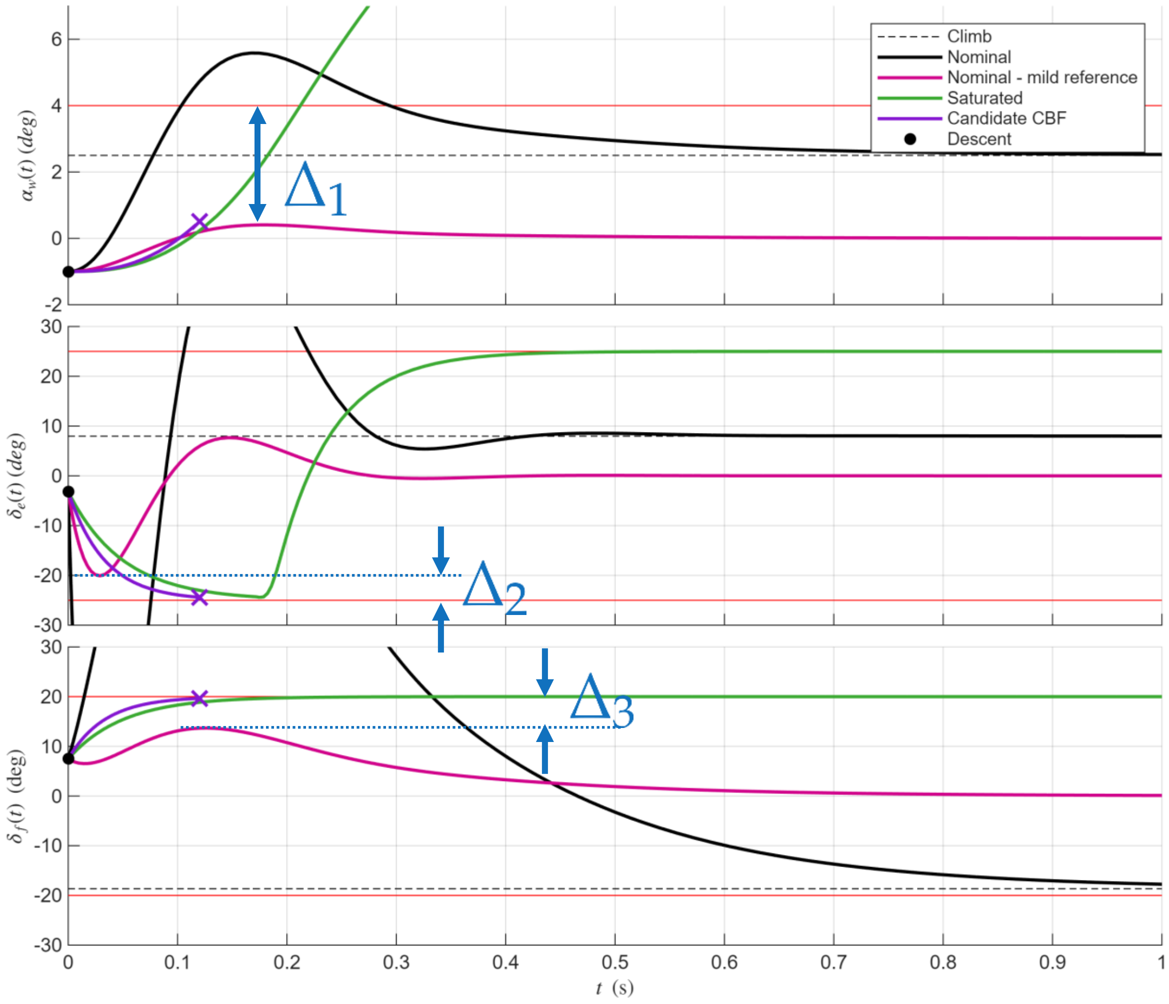

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Theorem: *DSMs are CBFs*

If Δ is a DSM, then it is a CBF for an augmented system $\begin{bmatrix} \boldsymbol{x} & v \end{bmatrix}^\top$.

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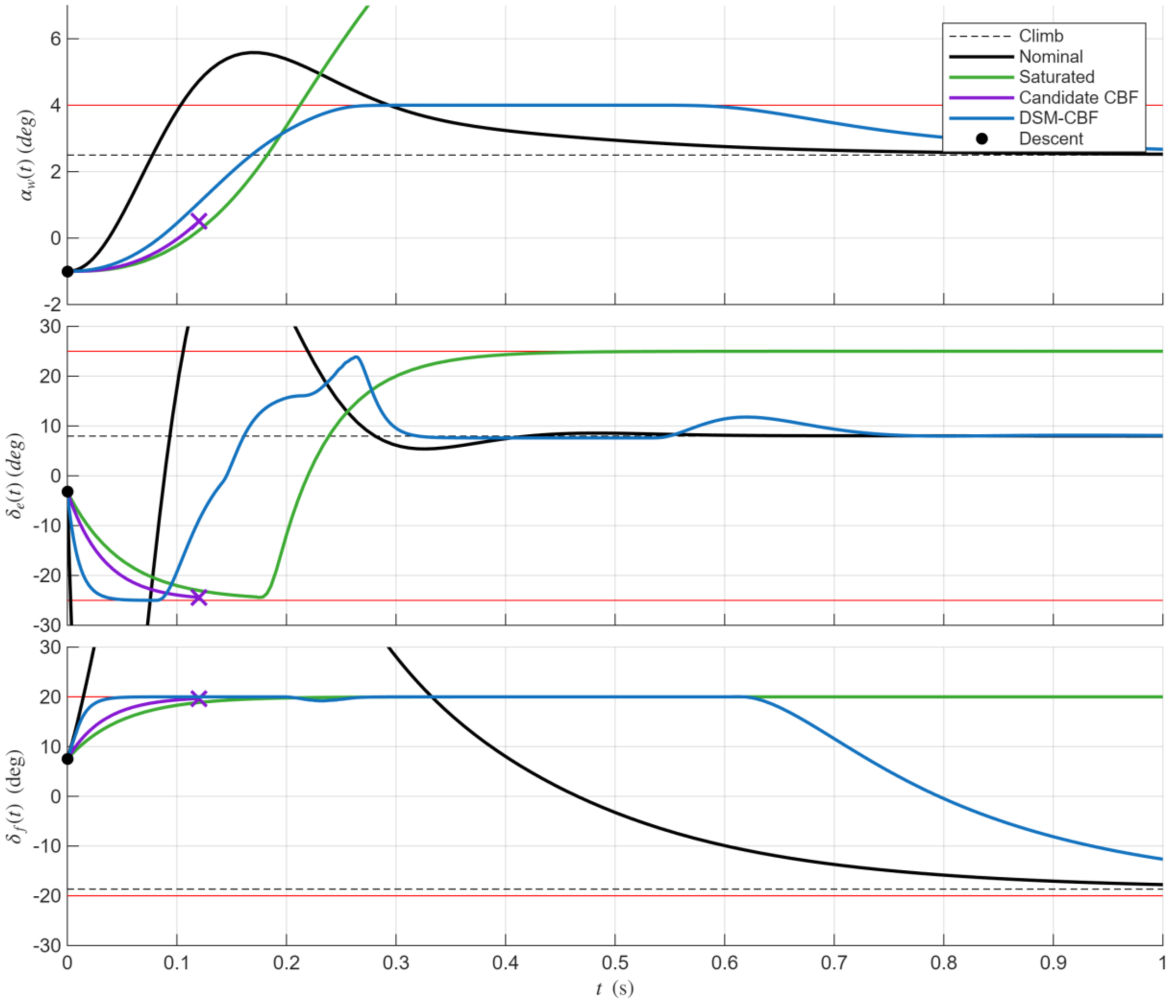

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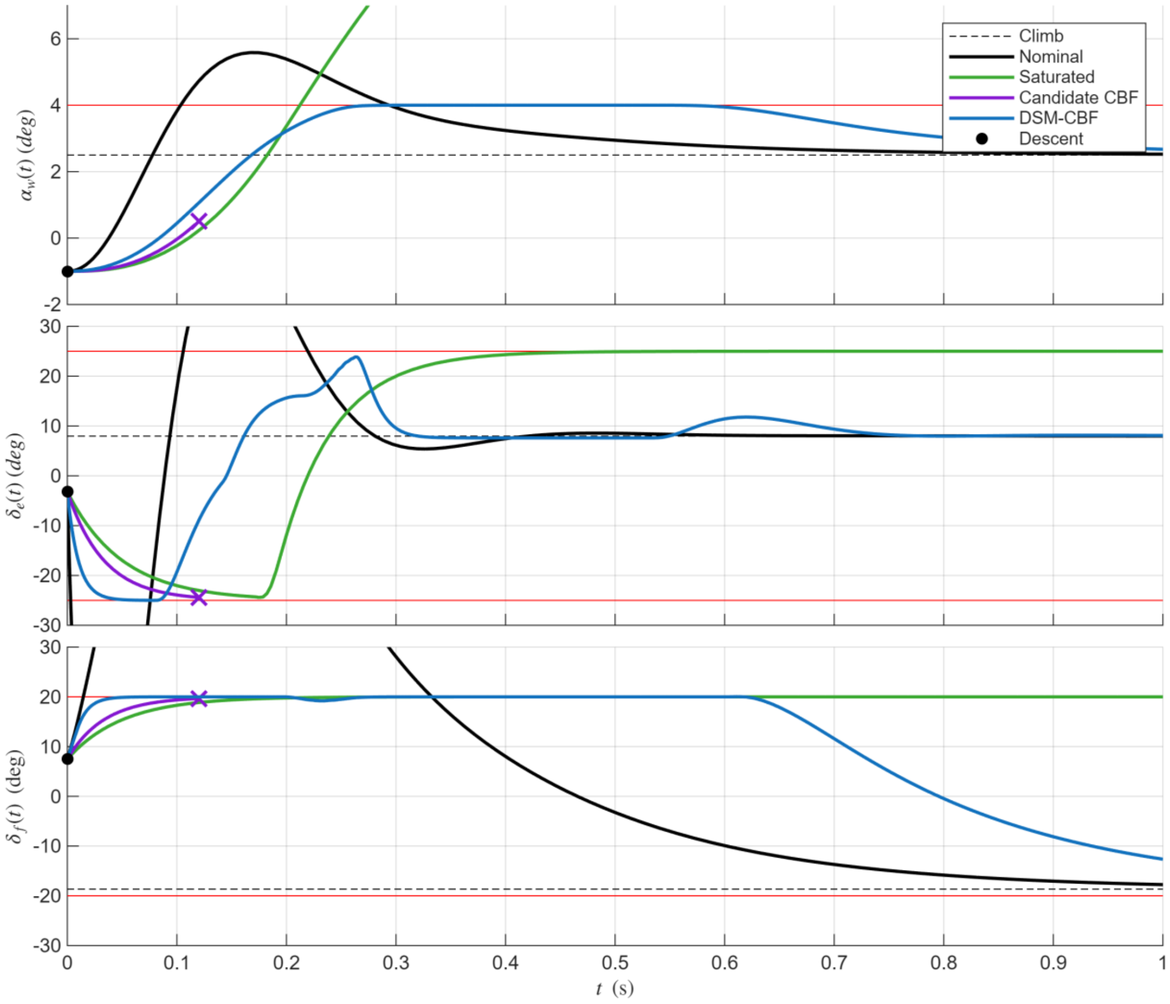

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mosek

DSM-CBF: 16 ms solvetime

Candidate CBF: 3.5 ms solvetime

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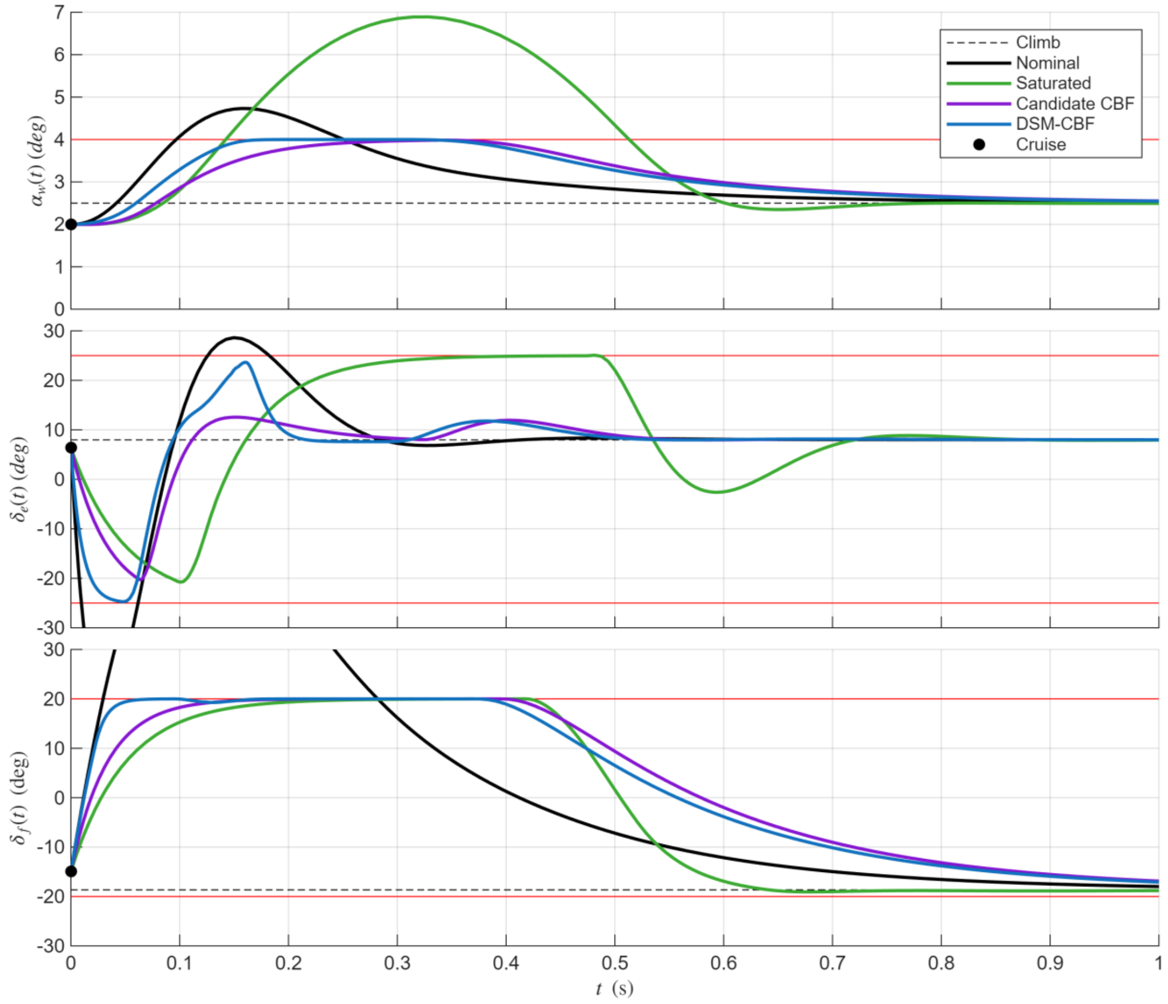

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