Safe GNC Design to Handle Polytopic Obstacles, Pointing Constraints, and Other Spacecraft

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Outline

1. Safe Control Problem
2. Light Background on CLFs and CBFs
3. How to Construct CBFs
4. Hybrid CLF-CBF Controller
5. CBF Construction from a CCG Observer

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1 Safe Control Problem

2 Light Background on CLFs and CBFs

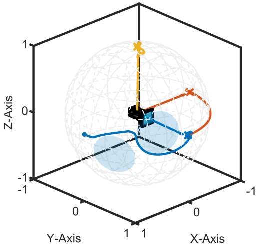
3 How to Construct CBFs

4 Hybrid CLF-CBF Controller

5 CBF Construction from a CCG Observer

Motivating Applications

- Consider safely reorienting a spacecraft
  - Possible keep-out regions to avoid pointing optical sensors to the sun;
  - Possible keep-in regions to guarantee star tracker functioning.
- Collision avoidance for in-orbit assembly;
- Docking with spacecraft having irregular bodies.



Safe Control Problem

- Consider a nonlinear control-affine system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x})\mathbf{u}, \tag{1}$$

- $\mathbf{x} \in \mathbb{R}^n$ is the system state;
- $\mathbf{u} \in \mathbb{R}^m$ is the control input.

Safe Control Problem

Design a controller $\mathbf{k} : \mathbb{R}^n \to \mathbb{R}^m$ such that the closed-loop solution φ satisfies:

- $\lim_{t\to\infty} \|\varphi(t) - \bar{\mathbf{x}}\| = 0$ (asymptotic stability)
- $\forall \mathbf{x}_0 \in \mathcal{C}$, we have $\varphi(t) \in \mathcal{C}$ for all $t \geq 0$ (forward invariance).

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Control Lyapunov Functions (CLFs)

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A continuously differentiable, proper, and positive-definite function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ around a point $\bar{\mathbf{x}}$ is a CLF for the system (1) if there exists a class-$\mathcal{K}$ function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\bar{\mathbf{x}}\}$,

$$\inf_{\mathbf{u} \in \mathbb{R}^m} [L_{\mathbf{f}} V(\mathbf{x}) + L_{\mathbf{G}} V(\mathbf{x})\mathbf{u}] < -\gamma(V(\mathbf{x})). \tag{2}$$

- A CLF V induces a pointwise set of control vectors

$$K_{\mathrm{CLF}}(\mathbf{x}) = \{\mathbf{u} \in \mathbb{R}^m : L_{\mathbf{f}} V(\mathbf{x}) + L_{\mathbf{G}} V(\mathbf{x})\mathbf{u} \leq -\gamma(V(\mathbf{x}))\}.$$

Control Barrier Functions (CBFs)

Control Barrier Functions (CBFs)

Let $\mathcal{C} \subset \mathbb{R}^n$ be the 0-superlevel set of a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ with $\nabla h(\mathbf{x}) \neq \mathbf{0}$ when $h(\mathbf{x}) = 0$. The function h is a (zeroing) CBF for system (1) if there exists an extended class-$\mathcal{K}_\infty$ function $\alpha : \mathbb{R} \to \mathbb{R}$ such that, for all $\mathbf{x} \in \mathbb{R}^n$,

$$\sup_{\mathbf{u}\in\mathbb{R}^m} [L_{\mathbf{f}} h(\mathbf{x}) + L_{\mathbf{G}} h(\mathbf{x})\mathbf{u}] > -\alpha(h(\mathbf{x})). \tag{3}$$

- A CBF h induces a pointwise set of control vectors

$$K_{\text{CBF}}(\mathbf{x}) = \{\mathbf{u} \in \mathbb{R}^m : L_{\mathbf{f}} h(\mathbf{x}) + L_{\mathbf{G}} h(\mathbf{x})\mathbf{u} \geq -\alpha(h(\mathbf{x}))\}.$$

Safety Filter QP

- We can use CBFs to enforce constraints on a pre-existing controller

$$\begin{aligned} \mathbf{k}(\mathbf{x}) = \underset{\mathbf{u}\in\mathbb{R}^m}{\arg\min}\ \frac{1}{2}\|\mathbf{u} - \mathbf{k}_{\mathsf{d}}(\mathbf{x})\|^2 \\ \text{subject to } \dot{h}(\mathbf{x},\mathbf{u}) \geq -\alpha(h(\mathbf{x})), \end{aligned} \tag{4}$$

- Using the Karush-Kuhn-Tucker (KKT) conditions, the solution is

$$\mathbf{k}(\mathbf{x}) = \mathbf{k}_{\mathsf{d}}(\mathbf{x}) + \mu(\mathbf{x}) L_{\mathbf{G}} h(\mathbf{x})^\top, \tag{5}$$

- Since we only have 1 constraint, the solution is given by two branches.

Safety Filter QP

- The two possible solutions are

$$\mu(\mathbf{x}) = \begin{cases} \bar{\mu}(\mathbf{x}), & \text{if } \dot{h}(\mathbf{x}, \mathbf{k}_{\mathsf{d}}(\mathbf{x})) \leq -\alpha(h(\mathbf{x})), \\ 0, & \text{if } \dot{h}(\mathbf{x}, \mathbf{k}_{\mathsf{d}}(\mathbf{x})) > -\alpha(h(\mathbf{x})), \end{cases} \tag{6}$$

- The correction to the unconstrained solution is given by

$$\bar{\mu}(\mathbf{x}) = -\frac{\dot{h}(\mathbf{x}, \mathbf{k}_{\mathsf{d}}(\mathbf{x})) + \alpha(h(\mathbf{x}))}{\|L_{\mathbf{G}} h(\mathbf{x})\|^2}. \tag{7}$$

- The overall controller is locally Lipschitz in $\mathbf{x}$ if all functions satisfy the same assumption.

CLF-CBF-QP

- Having a stable and safe controller means:

$$
\begin{aligned}
(\mathbf{k}(\mathbf{x}), \cdot) = \underset{(\mathbf{u},\delta)\in\mathbb{R}^{m+1}}{\arg\min} \ & \frac{1}{2}\|\mathbf{u}\|^2 + \frac{1}{2}p\delta^2 \\
\text{subject to } & L_{\mathbf{f}}V(\mathbf{x}) + L_{\mathbf{G}}V(\mathbf{x})\mathbf{u} \leq -\gamma(V(\mathbf{x})) + \delta, \\
& L_{\mathbf{f}}h(\mathbf{x}) + L_{\mathbf{G}}h(\mathbf{x})\mathbf{u} \geq -\alpha(h(\mathbf{x})),
\end{aligned} \tag{8}
$$

- We are minimizing actuation energy in a Quadratic Program (QP);
- Forcing the control vector to belong to the set $K_{\mathrm{CLF}}(\mathbf{x}) \cap K_{\mathrm{CBF}}(\mathbf{x})$;
- Parameter δ softens the stability requirement to allow feasibility.

Closed-form Solution based on the KKT conditions

- Define $F_V(\mathbf{x}) = L_\mathbf{f} V(\mathbf{x}) + \gamma(V(\mathbf{x}))$, $F_h(\mathbf{x}) = L_\mathbf{f} h(\mathbf{x}) + \alpha(h(\mathbf{x}))$, and $L(\mathbf{x}) = L_\mathbf{G} V(\mathbf{x}) L_\mathbf{G} h(\mathbf{x})^\top$
- Previous QP had 2 constraints so its solution has 4 branches.
- Optimal solution corresponds to

$$\mathbf{k}(\mathbf{x}) = \begin{cases} \mathbf{k}_1(\mathbf{x}), & \text{if } \mathbf{x} \in \mathcal{S}_1, \\ \mathbf{k}_2(\mathbf{x}), & \text{if } \mathbf{x} \in \mathcal{S}_2, \\ \mathbf{k}_3(\mathbf{x}), & \text{if } \mathbf{x} \in \mathcal{S}_3, \\ \mathbf{0}, & \text{if } \mathbf{x} \in \mathcal{S}_4, \end{cases} \tag{9}$$

Closed-form Solution based on the KKT conditions

- We can explicitly compute the expression for each branch:

$$\begin{aligned}
\mathbf{k}_1(\mathbf{x}) &= -\left(p^{-1} + \|L_{\mathbf{G}}V(\mathbf{x})\|^2\right)^{-1} F_V(\mathbf{x}) L_{\mathbf{G}}V(\mathbf{x})^\top, \\
\mathbf{k}_2(\mathbf{x}) &= -\|L_{\mathbf{G}}h(\mathbf{x})\|^{-2} F_h(\mathbf{x}) L_{\mathbf{G}}h(\mathbf{x})^\top, \\
\mathbf{k}_3(\mathbf{x}) &= -\lambda_1(\mathbf{x}) L_{\mathbf{G}}V(\mathbf{x})^\top + \lambda_2(\mathbf{x}) L_{\mathbf{G}}h(\mathbf{x})^\top.
\end{aligned} \tag{10}$$

- where

$$\lambda_1(\mathbf{x}) = \Delta(\mathbf{x})^{-1}\left(L(\mathbf{x})F_h(\mathbf{x}) - \|L_{\mathbf{G}}h(\mathbf{x})\|^2 F_V(\mathbf{x})\right),$$

$$\lambda_2(\mathbf{x}) = \Delta(\mathbf{x})^{-1}\left(\left(p^{-1} + \|L_{\mathbf{G}}V(\mathbf{x})\|^2\right) F_h(\mathbf{x}) - L(\mathbf{x})F_V(\mathbf{x})\right),$$

$$\Delta(\mathbf{x}) = L(\mathbf{x})^2 - \left(p^{-1} + \|L_{\mathbf{G}}V(\mathbf{x})\|^2\right)\|L_{\mathbf{G}}h(\mathbf{x})\|^2$$

- The important feature is that these algebraic expressions are constant for a given $\mathbf{x}$!

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Safe Reorientation with CBFs

- Let us write the dynamics in Euler angles

$$\dot{\boldsymbol{\eta}} = \mathbf{T}(\boldsymbol{\eta})^{-1}\boldsymbol{\omega}, \tag{11}$$

$$\dot{\boldsymbol{\omega}} = \mathbf{J}^{-1}\mathbf{S}(\mathbf{J}\boldsymbol{\omega})\boldsymbol{\omega} + \mathbf{J}^{-1}\boldsymbol{\tau}, \tag{12}$$

- Slight abuse of notation since there are singularities with $\mathbf{T}(\boldsymbol{\eta})^{-1}$

$$\mathbf{T}(\boldsymbol{\eta}) = \begin{bmatrix} 1 & 0 & -\sin(\theta) \\ 0 & \cos(\phi) & \sin(\phi)\cos(\theta) \\ 0 & -\sin(\phi) & \cos(\phi)\cos(\theta) \end{bmatrix} \tag{13}$$

- Makes the system easier to present in control-affine format.

CLF Design for Safe Reorientation

- Since we want to drive to desired Euler angles $\bar{\boldsymbol{\eta}}$

$$V_0(\boldsymbol{\eta}) = \frac{1}{2}\|\boldsymbol{\eta} - \bar{\boldsymbol{\eta}}\|^2 \tag{14}$$

- For the backstepping design, let us use

$$\mathbf{k}_{V_0}(\boldsymbol{\eta}) = -\frac{\bar{\gamma}_0}{2}\mathbf{T}(\boldsymbol{\eta})(\boldsymbol{\eta} - \bar{\boldsymbol{\eta}}) \tag{15}$$

- The total CLF for the backstepping procedure is

$$V(\boldsymbol{\eta}, \boldsymbol{\omega}) = V_0(\boldsymbol{\eta}) + \frac{1}{2\mu_V}\|\boldsymbol{\omega} - \mathbf{k}_{V_0}(\boldsymbol{\eta})\|^2 \tag{16}$$

CBF Design for Pointing Constraints

- We can trivially select CBF candidates

$$h_{\eta,0}^{(i)}(\boldsymbol{\eta}) = \cos(\theta_i) - \mathbf{d}_i^\top \mathbf{R}_\eta(\boldsymbol{\eta})\mathbf{v}_i^b \tag{17}$$

- The safe set would correspond to taking the minimum of candidate CBFs

$$\bigcap_{i \in \mathcal{I}} \mathcal{C}_{\eta,0}^{(i)} = \{\boldsymbol{\eta} \in \mathbb{R}^3 : \min_{i \in \mathcal{I}} h_{\eta,0}^{(i)}(\boldsymbol{\eta}) \geq 0\}. \tag{18}$$

- The minimum function is not differentiable but we can use the LogSumExp approximation

$$h_{\eta,0}(\boldsymbol{\eta}) = -\frac{1}{\kappa_0} \ln\left(\sum_{i \in \mathcal{I}} \exp\left(-\kappa_0 h_{\eta,0}^{(i)}(\boldsymbol{\eta})\right)\right) \tag{19}$$

CBF Design for Pointing Constraints

- In order to check that it is indeed a valid CBF, take its time derivative

$$\dot{h}_{\eta,0}(\boldsymbol{\eta}, \boldsymbol{\omega}) = \nabla h_{\eta,0}(\boldsymbol{\eta})^\top \mathbf{T}(\boldsymbol{\eta})^{-1} \boldsymbol{\omega} \tag{20}$$

- Since $\nabla h_{\eta,0}(\boldsymbol{\eta}) \neq \mathbf{0}$ at the boundary of the safe set, then

$$\sup_{\boldsymbol{\omega} \in \mathbb{R}^3} \dot{h}_{\eta,0}(\boldsymbol{\eta}, \boldsymbol{\omega}) > -\alpha_{\eta,0}(h_{\eta,0}(\boldsymbol{\eta})), \tag{21}$$

- Similarly to the CLF design, we need a controller for the top subsystem

$$\mathbf{k}_{h_{\eta,0}}(\boldsymbol{\eta}) = \bar{\alpha}_{\eta,0} \mathbf{T}(\boldsymbol{\eta}) \nabla h_{\eta,0}(\boldsymbol{\eta}) \tag{22}$$

CBF Design for Pointing Constraints

- We can check back what happens to the time derivative

$$\dot{h}_{\eta,0}(\boldsymbol{\eta}, \mathbf{k}_{h_{\eta,0}}(\boldsymbol{\eta})) = \bar{\alpha}_{\eta,0}\|\nabla h_{\eta,0}(\boldsymbol{\eta})\|^2. \quad (23)$$

- For the backstepping procedure, the total CBF is given by

$$h_\eta(\boldsymbol{\eta}, \boldsymbol{\omega}) = h_{\eta,0}(\boldsymbol{\eta}) - \frac{1}{2\mu_h}\|\boldsymbol{\omega} - \mathbf{k}_{h_{\eta,0}}(\boldsymbol{\eta})\|^2 \quad (24)$$

- Therefore, it is guaranteed that selecting an appropriate constant $\bar{\alpha}_{\eta,0} \in \mathbb{R}_{>0}$ makes

$$\sup_{\boldsymbol{\tau} \in \mathbb{R}^3} \dot{h}_\eta(\boldsymbol{\eta}, \boldsymbol{\omega}, \boldsymbol{\tau}) > -\alpha_\eta(h_\eta(\boldsymbol{\eta}, \boldsymbol{\omega})), \quad (25)$$

CBF Design for Maximum Angular Rates

- Instead of using the infinity norm, we can use a lower-order p-norm

$$h_\omega(\boldsymbol{\eta}, \boldsymbol{\omega}) = \frac{1}{p}\omega_{\mathsf{max}}^p - \frac{1}{p}\|\boldsymbol{\omega}\|_p^p \tag{26}$$

- Its time derivative is

$$\dot{h}_\omega(\boldsymbol{\eta}, \boldsymbol{\omega}, \boldsymbol{\tau}) = \nabla_{\boldsymbol{\omega}} h_\omega(\boldsymbol{\eta}, \boldsymbol{\omega})^\top (\mathbf{J}^{-1}\mathbf{S}(\mathbf{J}\boldsymbol{\omega})\boldsymbol{\omega} + \mathbf{J}^{-1}\boldsymbol{\tau}) \tag{27}$$

- We can also assert

$$\sup_{\boldsymbol{\tau}\in\mathbb{R}^3} \dot{h}_\omega(\boldsymbol{\eta}, \boldsymbol{\omega}, \boldsymbol{\tau}) > -\alpha_\omega(h_\omega(\boldsymbol{\eta}, \boldsymbol{\omega})), \tag{28}$$

Finalizing the CBF Design

- We could just solve a QP with all the CBFs as constraints.
- But we can resort to the same LogSumExp trick and use:

$$h(\boldsymbol{\eta},\boldsymbol{\omega}) = -\frac{1}{\kappa}\ln\left(\exp\left(-\kappa h_\eta(\boldsymbol{\eta},\boldsymbol{\omega})\right) + \exp\left(-\kappa h_\omega(\boldsymbol{\eta},\boldsymbol{\omega})\right)\right) \quad (29)$$

- Final controller corresponds to

$$\begin{aligned}(\mathbf{k}^*(\boldsymbol{\eta},\boldsymbol{\omega}),\cdot) = \underset{(\boldsymbol{\tau},\delta)\in\mathbb{R}^4}{\arg\min}\ & \frac{1}{2}(\|\boldsymbol{\tau}\|^2 + \rho\delta^2) \\ \text{subject to } & \dot{V}(\boldsymbol{\eta},\boldsymbol{\omega},\boldsymbol{\tau}) \leq -\bar{\gamma}V(\boldsymbol{\eta},\boldsymbol{\omega}) + \delta, \\ & \dot{h}(\boldsymbol{\eta},\boldsymbol{\omega},\boldsymbol{\tau}) \geq -\bar{\alpha}h(\boldsymbol{\eta},\boldsymbol{\omega}),\end{aligned} \quad (30)$$

CLF-CBF Controller Performance

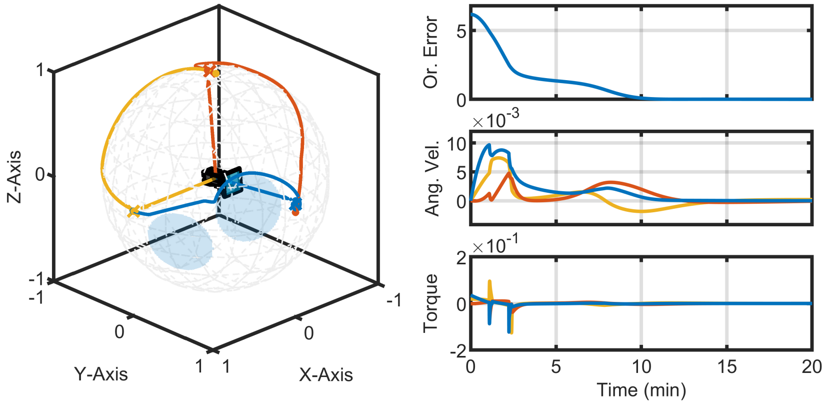
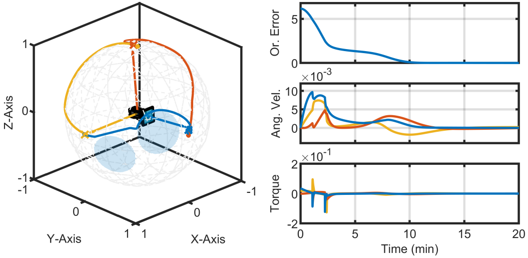


Figure: Safe Reorientation from large attitude error.

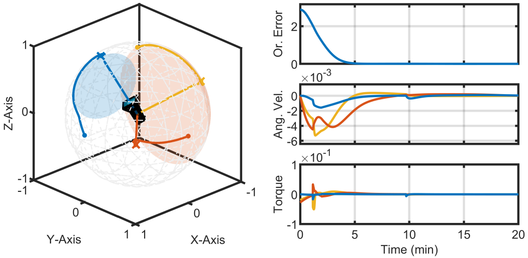
CLF-CBF Controller Performance

- It is possible to reorient from large error attitude.
- Input constraints are implicitly encoded by selecting appropriate constants.



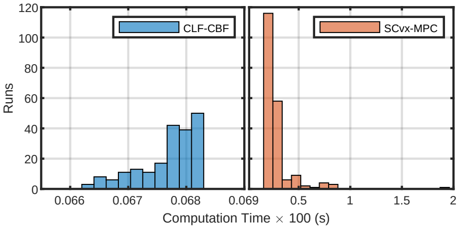
CLF-CBF Controller Performance

- Controller can handle large angles for the pointing constraints.
- Control law produces smooth torques to avoid sloshing.



CLF-CBF Controller *vs* SCvx + MPC

- SCvx + MPC is worse in terms of computation complexity.
- The method also does not guarantee feasibility of the trajectory (can violate the constraints).
- SCvx + MPC is an order of magnitude slower.



Take-home Message

- By exploiting the bounded nature of the angles, CBFs can be designed in a principled way.
- Multiple CBFs can be grouped using the LogSumExp to maintain computational efficiency.
- Backstepping addresses the issue of higher relative degree for the dynamics.

Take-home Message

- By exploiting the bounded nature of the angles, CBFs can be designed in a principled way.
- Multiple CBFs can be grouped using the LogSumExp to maintain computational efficiency.
- Backstepping addresses the issue of higher relative degree for the dynamics.

Key Issue

From Hybrid Systems theory, this type of controllers cannot be globally asymptotically stable.

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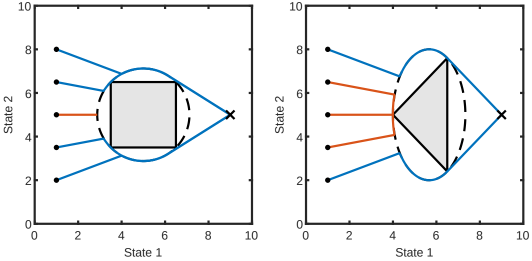
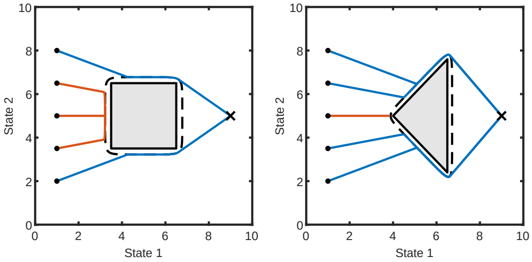
Undesirable Stable Equilibria

- A continuous controller cannot be globally asymptotic stable in the presence of obstacles.
- For a single-integrator dynamics and ellipsoidal unsafe sets, we get undesired equilibria.
- This strategy is conservative for polytopic constraints.



Undesirable Stable Equilibria

- If we use again the LogSumExp trick to improve conservativeness.
- We still get undesirable equilibria.
- The region of attraction might not be a single point.



Hybrid CLF-CBF Controllers

- The approach to resolve this issue is to adopt a Hybrid controller.
- The state is augmented with the logic state vector ξ evolving in flow

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x})\mathbf{k}_{\xi}(\mathbf{x}) \\ \mathbf{0} \end{bmatrix}, \quad (\mathbf{x}, \xi) \in \mathcal{F}, \tag{31}$$

- The jump dynamics (i.e., the logic implementation) is

$$\begin{bmatrix} \mathbf{x}^{+} \\ \xi^{+} \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ \mathbf{s}(\mathbf{x}, \xi) \end{bmatrix}, \quad (\mathbf{x}, \xi) \in \mathcal{J}. \tag{32}$$

Switching Logic

- The decision vector $\boldsymbol{\xi}$ contains
  - $\hat{\mathbf{x}}$ - the current desired target for the state;
  - q - identifies which hyperplane is active through the CBF

$$h_q(\mathbf{x}) = \mathbf{n}_q^\top \mathbf{x} - d_q. \tag{33}$$

  - The last hyperplane to be active satisfies

$$\bar{q} = \underset{q' \in \{1,\ldots,Q\}}{\arg\max}\ h_{q'}(\bar{\mathbf{x}}). \tag{34}$$

  - The next safe halfspace is selected based on

$$\hat{q} = \underset{\left\{q' \in \{1,\ldots,Q\} : \mathbf{v}^\top \mathbf{n}_{q'} > \mathbf{v}^\top \mathbf{n}_q\right\} \cup \{\bar{q}\}}{\arg\max} h_{q'}(\hat{\mathbf{x}}), \tag{35}$$

Switching Logic

- Building on the concept of synergistic Lyapunov function, we can force a hysteresis

$$\begin{aligned} \mathcal{F} &= \{(\mathbf{x}, \boldsymbol{\xi}) \in \mathcal{H} : h_{\hat{q}}(\mathbf{x}) - h_q(\mathbf{x}) < \sigma \vee h_q(\mathbf{x}) < 0\}, \\ \mathcal{J} &= \{(\mathbf{x}, \boldsymbol{\xi}) \in \mathcal{H} : h_{\hat{q}}(\mathbf{x}) - h_q(\mathbf{x}) \geq \sigma \wedge h_q(\mathbf{x}) \geq 0\}, \end{aligned} \tag{36}$$

- The jump set requires the state to be inside the next safe set.
- The definition of new safe halfspace prevents switching in a cycle.
- The next $\hat{\mathbf{x}}$ can be selected inside the new safe halfspace.

Controller Feasibility

- The previous hybrid design does not guarantee that the resulting QP will be feasible.
- The intersection of the control half-spaces can only be empty when

$$\mathcal{S}_{\boldsymbol{\xi}}^{c} = \{\mathbf{x} \in \mathbb{R}^n : \exists \lambda \in \mathbb{R}_{>0} : L_{\mathbf{G}} V_{\hat{\mathbf{x}}}(\mathbf{x}) = \lambda L_{\mathbf{G}} h_q(\mathbf{x}), \\ L_{\mathbf{G}} V_{\hat{\mathbf{x}}}(\mathbf{x}), L_{\mathbf{G}} h_q(\mathbf{x}) \neq \mathbf{0}\}, \tag{37}$$

- The first condition translates to an alignment

$$(\nabla V_{\hat{\mathbf{x}}}(\mathbf{x}) - \lambda \nabla h_q(\mathbf{x}))^{\top} \mathbf{G}(\mathbf{x}) = \mathbf{0}. \tag{38}$$

- And $\lambda = \frac{\|\nabla V_{\hat{\mathbf{x}}}(\mathbf{x})\|}{\|\nabla h_q(\mathbf{x})\|}$.

Controller Feasibility

- Plugging the value of λ, we get

$$-\alpha(h_q(\mathbf{x})) \leq L_{\mathbf{f}} h_q(\mathbf{x}) + L_{\mathbf{G}} h_q(\mathbf{x})\mathbf{u} \leq -\lambda^{-1}\gamma(V_{\hat{\mathbf{x}}}(\mathbf{x})) \quad (39)$$

- Meaning that the CLF and CBF are compatible when $\mathbf{x} \in \mathcal{S}^{\mathrm{c}}_{\boldsymbol{\xi}}$ if we select the functions

$$\alpha(h_q(\mathbf{x})) \geq \frac{\|\nabla h_q(\mathbf{x})\|}{\|\nabla V_{\hat{\mathbf{x}}}(\mathbf{x})\|}\gamma(V_{\hat{\mathbf{x}}}(\mathbf{x})). \quad (40)$$

Controller Feasibility

- Plugging the value of λ, we get

$$-\alpha(h_q(\mathbf{x})) \leq L_{\mathbf{f}} h_q(\mathbf{x}) + L_{\mathbf{G}} h_q(\mathbf{x})\mathbf{u} \leq -\lambda^{-1}\gamma(V_{\hat{\mathbf{x}}}(\mathbf{x})) \quad (39)$$

- Meaning that the CLF and CBF are compatible when $\mathbf{x} \in \mathcal{S}_{\boldsymbol{\xi}}^{c}$ if we select the functions

$$\alpha(h_q(\mathbf{x})) \geq \frac{\|\nabla h_q(\mathbf{x})\|}{\|\nabla V_{\hat{\mathbf{x}}}(\mathbf{x})\|}\gamma(V_{\hat{\mathbf{x}}}(\mathbf{x})). \quad (40)$$

Compatibility condition

Select $\gamma(s) = 2\bar{\gamma}s$ and $\alpha(s) = \bar{\alpha}s$ with $\bar{\alpha} \geq \bar{\gamma}$.

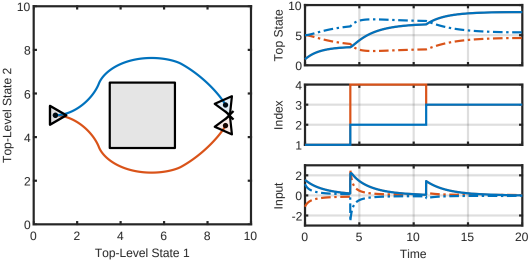
Handling Input Constraints

- For the attitude application, we could compute the parameters to satisfy the input constraints.
- However, we do not have bounded domain in the general setting.
- Adopt the optimal-decay QP formulation:

$$\begin{aligned}(\mathbf{k}_{\boldsymbol{\xi}}(\mathbf{x}), \cdot) = \underset{(\mathbf{u},\omega)\in\mathbb{R}^{m+1}}{\arg\min}\ & \frac{1}{2}\|\mathbf{u}\|^2 + \frac{1}{2}p(\omega-1)^2 \\ \text{subject to}\ & L_{\mathbf{f}}V_{\hat{\mathbf{x}}}(\mathbf{x}) + L_{\mathbf{G}}V_{\hat{\mathbf{x}}}(\mathbf{x})\mathbf{u} \leq -\gamma(V_{\hat{\mathbf{x}}}(\mathbf{x})) + \delta_{\hat{\mathbf{x}}}(\mathbf{x}), \\ & L_{\mathbf{f}}h_q(\mathbf{x}) + L_{\mathbf{G}}h_q(\mathbf{x})\mathbf{u} \geq -\omega\alpha(h_q(\mathbf{x})), \\ & \|\mathbf{u}\| \leq u_{\max},\end{aligned} \tag{41}$$

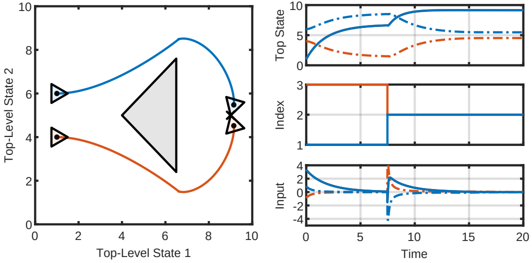
Simulation with Unicycle Dynamics

- Notice that the unicycle has mixed relative degrees.
- However, we can control a point located ahead of the center of mass.
- The hybrid CLF-CBF controller does not have undesirable equilibria.



Simulation with Unicycle Dynamics

- The separation from the obstacle boundary depends on the synergistic parameter.
- Closed-loop exhibits an asymptotic convergence to the intermediate points.
- There is a finite number of jumps.

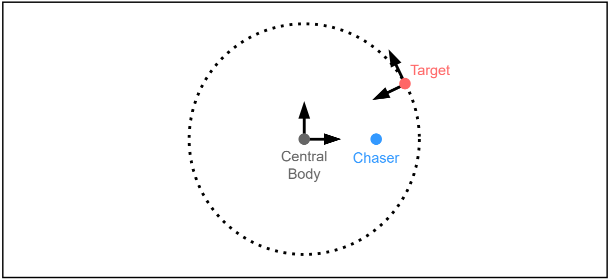


Safe Spacecraft Rendezvous

- In LVLH frame centered at the target, chaser dynamics are

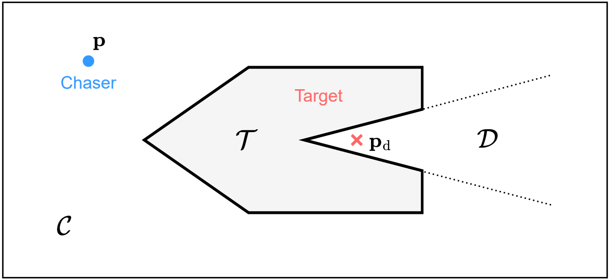
$$\dot{\mathbf{p}} = \mathbf{v},$$

$$\dot{\mathbf{v}} = -\mathbf{S}(\omega)^2\mathbf{p} - 2\mathbf{S}(\omega)\mathbf{v} - \omega^2 R_{\mathsf{T}}\mathbf{e}_2 - \frac{Gm_{\mathsf{E}}}{\|\mathbf{R}(t)\mathbf{p} + \varphi_{\mathsf{T}}(t)\|^3}\left(\mathbf{p} + \mathbf{R}(t)^\top \varphi_{\mathsf{T}}(t)\right) + \frac{1}{m_{\mathsf{C}}}\mathbf{u}, \tag{42}$$



Rendezvous and Docking

- Due to the nonconvex shape of the target, we split the mission in:
  - Rendezvous phase - safe maneuver to enter in $\mathcal{D}$;
  - Docking phase - safe approach to the desired position $\mathbf{p}_\mathrm{d}$.



Rendezvous Controller Design

- We can define the convex hull of the target body as the polytope

$$\mathcal{P} = \{\mathbf{p} \in \mathbb{R}^p : h_1(\mathbf{p}) < 0 \wedge \cdots \wedge h_Q(\mathbf{p}) < 0\}, \tag{43}$$

- Following the Hybrid CLF-CBF to obtain the QP:

$$\begin{aligned} \mathbf{k}_{\boldsymbol{\xi},1}(\mathbf{p},\mathbf{v},t) = \operatorname*{arg\,min}_{\mathbf{u}\in\mathbb{R}^m} \ & \frac{1}{2}\|\mathbf{u} - \mathbf{k}_{\mathsf{n}}(\mathbf{p},\mathbf{v},t)\|^2 \\ \text{subject to } & (L_{\bar{\mathbf{f}}_1} V_{\hat{\mathbf{p}},1})(\mathbf{p},\mathbf{v},t) + (L_{\bar{\mathbf{G}}_1} V_{\hat{\mathbf{p}},1})(\mathbf{p},\mathbf{v},t)\mathbf{u} \leq -\gamma_1\left(V_{\hat{\mathbf{p}},1}(\mathbf{p},\mathbf{v})\right), \\ & (L_{\bar{\mathbf{f}}_1} h_{q,1})(\mathbf{p},\mathbf{v},t) + (L_{\bar{\mathbf{G}}_1} h_{q,1})(\mathbf{p},\mathbf{v},t)\mathbf{u} \geq -\alpha_1\left(h_{q,1}(\mathbf{p},\mathbf{v})\right), \end{aligned} \tag{44}$$

Docking Controller Design

- For docking, we must only satisfy the constraints defining $\mathcal{D}$

$$\mathcal{D} = \{\mathbf{p} \in \mathbb{R}^p : \psi_1(\mathbf{p}) \geq 0 \wedge \cdots \wedge \psi_M(\mathbf{p}) \geq 0\}\,, \tag{45}$$

- Since this is enforcing belonging to a convex set, we can use the LogSumExp approximation

$$h_{\mathsf{d}}(\mathbf{p}) = -\frac{1}{\kappa}\ln\left(\sum_{i=1}^{M}\exp(-\kappa\psi_i(\mathbf{p}))\right) \tag{46}$$

- Following the Hybrid CLF-CBF to obtain the QP:

$$\begin{aligned}\mathbf{k}_{\mathsf{d},1}(\mathbf{p},\mathbf{v},t) = \underset{\mathbf{u}\in\mathbb{R}^m}{\arg\min}\ & \frac{1}{2}\|\mathbf{u} - \mathbf{k}_{\mathsf{n}}(\mathbf{p},\mathbf{v},t)\|^2 \\ \text{subject to}\ & (L_{\bar{\mathbf{f}}_1}V_{\mathsf{d},1})(\mathbf{p},\mathbf{v},t) + (L_{\bar{\mathbf{G}}_1}V_{\mathsf{d},1})(\mathbf{p},\mathbf{v},t)\mathbf{u} \leq -\gamma_1\left(V_{\mathsf{d},1}(\mathbf{p},\mathbf{v})\right), \\ & (L_{\bar{\mathbf{f}}_1}h_{\mathsf{d},1})(\mathbf{p},\mathbf{v},t) + (L_{\bar{\mathbf{G}}_1}h_{\mathsf{d},1})(\mathbf{p},\mathbf{v},t)\mathbf{u} \geq -\alpha_1\left(h_{\mathsf{d},1}(\mathbf{p},\mathbf{v})\right),\end{aligned} \tag{47}$$

Simulation of Safe Rendezvous and Docking

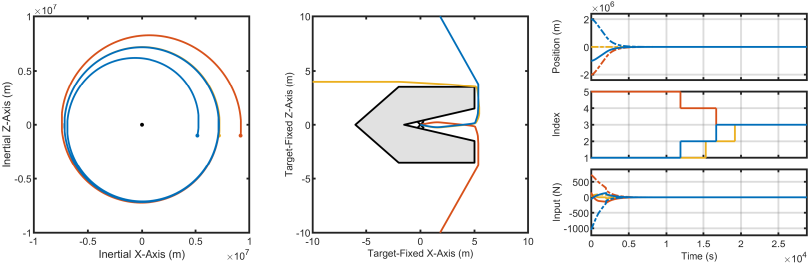


Figure: Starting in the same orbit behind the target.

Simulation of Safe Rendezvous and Docking

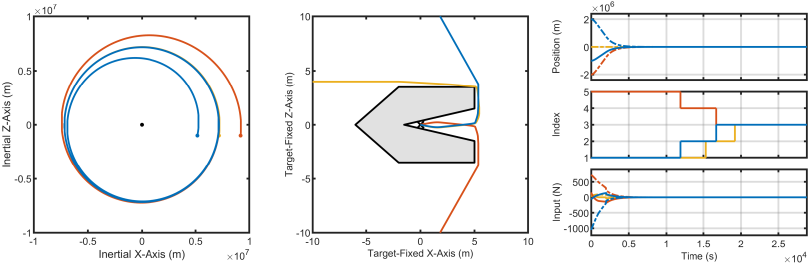
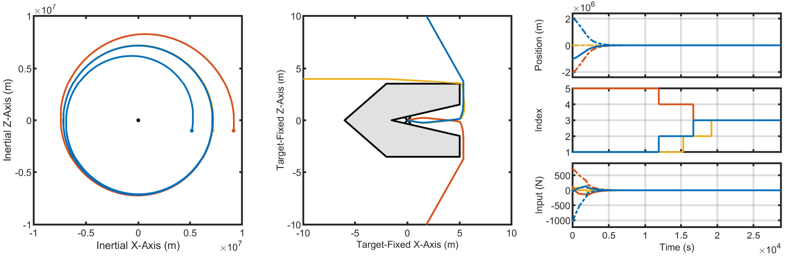


Figure: Starting in a higher orbit.

Simulation of Safe Rendezvous and Docking

- We use the nominal controller which minimizes acceleration of the chaser

$$\mathbf{k}_{\mathsf{n}}(\mathbf{p}, \mathbf{v}, t) = m_{\mathsf{C}} \mathbf{S}(\omega)^2 \mathbf{p} + 2 m_{\mathsf{C}} \mathbf{S}(\omega) \mathbf{v} + m_{\mathsf{C}} \omega^2 R_{\mathsf{T}} \mathbf{e}_2 + \frac{G m_{\mathsf{E}} m_{\mathsf{C}}}{\|\mathbf{R}(t)\mathbf{p} + \varphi_{\mathsf{T}}(t)\|^3} \left(\mathbf{p} + \mathbf{R}(t)^\top \varphi_{\mathsf{T}}(t)\right), \tag{48}$$



Take-home Message

- The Hybrid CLF-CBF controller can render the closed-loop globally asymptotically stable.
- Interesting features:
  - The control law is a closed-form expression.
  - We can check and design compatible CLF and CBF.

Take-home Message

- The Hybrid CLF-CBF controller can render the closed-loop globally asymptotically stable.
- Interesting features:
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  - We can check and design compatible CLF and CBF.

Main challenge

The safety definition must be known *a priori* at design stage.

Outline

1 Safe Control Problem

2 Light Background on CLFs and CBFs

3 How to Construct CBFs

4 Hybrid CLF-CBF Controller

5 CBF Construction from a CCG Observer

Safe Control Problem

- Consider a nonlinear control-affine system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x})\mathbf{u}, \quad (49)$$

- $\mathbf{x} \in \mathbb{R}^n$ is the system state;
- $\mathbf{u} \in \mathbb{R}^m$ is the control input.

Safe Control Problem

Design a controller $\mathbf{k} : \mathbb{R}^n \to \mathbb{R}^m$ such that the closed-loop solution φ satisfies:

- $\lim_{t\to\infty} \|\varphi(t) - \bar{\mathbf{x}}\| = 0$ (asymptotic stability)
- $\forall \mathbf{x}_0 \in \mathcal{C}$, we have $\varphi(t) \in \mathcal{C}$ for all $t \geq 0$ (set forward invariance).

Safe Control Problem (updated)

- Consider a nonlinear control-affine system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x})\mathbf{u}, \tag{49}$$

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Safe Control Problem

Design a controller $\mathbf{k} : \mathbb{R}^n \to \mathbb{R}^m$ such that the closed-loop solution φ satisfies:

- $\lim_{t\to\infty} \|\varphi(t) - \bar{\mathbf{x}}\| = 0$ (asymptotic stability)
- $\forall \mathbf{x}_0 \in \mathcal{C}(t)$, we have $\varphi(t) \in \mathcal{C}(t)$ for all $t \geq 0$ (set-valued flow forward invariance).

Control Barrier Functions (CBFs)

Control Barrier Functions (CBFs)

Let $\mathcal{C} \subset \mathbb{R}^n$ be the 0-superlevel set of a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ with $\nabla h(\mathbf{x}) \neq \mathbf{0}$ when $h(\mathbf{x}) = 0$. The function h is a (zeroing) CBF for system (1) if there exists an extended class-$\mathcal{K}_\infty$ function $\alpha : \mathbb{R} \to \mathbb{R}$ such that, for all $\mathbf{x} \in \mathbb{R}^n$,

$$\sup_{\mathbf{u}\in\mathbb{R}^m} [L_{\mathbf{f}} h(\mathbf{x}) + L_{\mathbf{G}} h(\mathbf{x})\mathbf{u}] > -\alpha(h(\mathbf{x})). \tag{50}$$

- A CLF V induces a pointwise set of control vectors

$$K_{\mathrm{CBF}}(\mathbf{x}) = \{\mathbf{u} \in \mathbb{R}^m : L_{\mathbf{f}} h(\mathbf{x}) + L_{\mathbf{G}} h(\mathbf{x})\mathbf{u} \geq -\alpha(h(\mathbf{x}))\}.$$

Control Barrier Functions (CBFs) (updated)

Control Barrier Functions (CBFs)

Let $\mathcal{C} : [t_0, t_f) \rightrightarrows \mathbb{R}^n$ be a set-valued flow for a continuously differentiable function $h : \mathbb{R}^n \times [t_0, t_f) \to \mathbb{R}$ with $\frac{\partial}{\partial \mathbf{x}} h(\mathbf{x}, t) \neq \mathbf{0}$ when $h(\mathbf{x}, t) = 0$. The function h is a (zeroing) CBF for system (1) if there exists an extended class-$\mathcal{K}_\infty$ function $\alpha : \mathbb{R} \to \mathbb{R}$ such that, for all $(\mathbf{x}, t) \in \mathcal{G}(\mathcal{D})$,

$$\sup_{\mathbf{u} \in \mathbb{R}^m} \left[L_\mathbf{f} h(\mathbf{x}, t) + L_\mathbf{G} h(\mathbf{x}, t)\mathbf{u} + \frac{\partial h(\mathbf{x}, t)}{\partial t}\right] > -\alpha(h(\mathbf{x}, t)). \quad (51)$$

- A CBF h induces a pointwise set of control vectors

$$K_{\mathrm{CBF}}(\mathbf{x}, t) = \{\mathbf{u} \in \mathbb{R}^m : \dot{h}(\mathbf{x}, t, \mathbf{u}) \geq -\alpha(h(\mathbf{x}, t))\}.$$

Safety Filter QP

- We can use CBFs to enforce constraints on a pre-existing controller

$$\begin{aligned} \mathbf{k}(\mathbf{x}) = \underset{\mathbf{u}\in\mathbb{R}^m}{\arg\min}\ & \frac{1}{2}\|\mathbf{u} - \mathbf{k}_{\mathsf{d}}(\mathbf{x})\|^2 \\ \text{subject to } & \dot{h}(\mathbf{x}, \mathbf{u}) \geq -\alpha(h(\mathbf{x})), \end{aligned} \tag{52}$$

- Using the Karush-Kuhn-Tucker (KKT) conditions, the solution is

$$\mathbf{k}(\mathbf{x}) = \mathbf{k}_{\mathsf{d}}(\mathbf{x}) + \mu(\mathbf{x}) L_{\mathbf{G}} h(\mathbf{x})^\top, \tag{53}$$

- Since we only have 1 constraint, the solution is given by two branches.

Safety Filter QP (updated)

- We can use CBFs to enforce constraints on a pre-existing controller

$$\mathbf{k}(\mathbf{x}, t) = \underset{\mathbf{u}\in\mathbb{R}^m}{\arg\min}\ \frac{1}{2}\|\mathbf{u} - \mathbf{k}_{\mathsf{d}}(\mathbf{x}, t)\|^2 \quad \text{subject to } \dot{h}(\mathbf{x}, t, \mathbf{u}) \geq -\alpha(h(\mathbf{x}, t)), \tag{54}$$

- Using the Karush-Kuhn-Tucker (KKT) conditions, the solution is

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- Since we only have 1 constraint, the solution is given by two branches.

Safety Filter QP

- The two possible solutions are

$$\mu(\mathbf{x}) = \begin{cases} \bar{\mu}(\mathbf{x}), & \text{if } \dot{h}(\mathbf{x}, \mathbf{k}_\mathsf{d}(\mathbf{x})) \leq -\alpha(h(\mathbf{x})), \\ 0, & \text{if } \dot{h}(\mathbf{x}, \mathbf{k}_\mathsf{d}(\mathbf{x})) > -\alpha(h(\mathbf{x})), \end{cases} \tag{56}$$

- The correction to the unconstrained solution is given by

$$\bar{\mu}(\mathbf{x}) = -\frac{\dot{h}(\mathbf{x}, \mathbf{k}_\mathsf{d}(\mathbf{x})) + \alpha(h(\mathbf{x}))}{\|L_{\mathbf{G}} h(\mathbf{x})\|^2}. \tag{57}$$

- The overall controller is locally Lipschitz in $\mathbf{x}$ if all functions satisfy the same assumption.

Safety Filter QP (updated)

- The two possible solutions are

$$\mu(\mathbf{x}, t) = \begin{cases} \bar{\mu}(\mathbf{x}, t), & \text{if } \dot{h}(\mathbf{x}, t, \mathbf{k}_\mathsf{d}(\mathbf{x}, t)) \leq -\alpha(h(\mathbf{x}, t)), \\ 0, & \text{if } \dot{h}(\mathbf{x}, t, \mathbf{k}_\mathsf{d}(\mathbf{x}, t)) > -\alpha(h(\mathbf{x}, t)), \end{cases} \tag{58}$$

- The correction to the unconstrained solution is given by

$$\bar{\mu}(\mathbf{x}, t) = -\frac{\dot{h}(\mathbf{x}, t, \mathbf{k}_\mathsf{d}(\mathbf{x}, t)) + \alpha(h(\mathbf{x}, t))}{\|L_\mathbf{G} h(\mathbf{x}, t)\|^2}. \tag{59}$$

- The overall controller is locally Lipschitz in $\mathbf{x}$ and continuous in t if all functions satisfy the same assumptions.

Safe Navigation Problem

- We will consider an ego vehicle described by

$$\begin{aligned} \dot{\mathbf{p}} &= \mathbf{f}(\mathbf{p}) + \mathbf{G}(\mathbf{p})\mathbf{z}, \\ \dot{\mathbf{z}} &= \mathbf{f}_1(\mathbf{p}, \mathbf{z}) + \mathbf{G}_1(\mathbf{p}, \mathbf{z})\mathbf{u}, \end{aligned} \tag{60}$$

- Ego vehicle has a body shape provided by

$$\mathcal{P}(\mathbf{p}) = \mathbf{p} + \bar{\mathcal{P}}, \tag{61}$$

- There are M dynamics obstacles with convex bodies

$$\mathcal{O}_i(\mathbf{o}_i) = \mathbf{o}_i + \bar{\mathcal{O}}_i, \tag{62}$$

Safe Navigation Problem

- Obstacles follow linear dynamics (our recent LCSS submission deals with uncertain nonlinear dynamics):

$$\begin{aligned} \dot{\mathbf{x}}_i &= \mathbf{F}_i \mathbf{x}_i + \mathbf{w}_i, \\ \mathbf{o}_i &= \mathbf{E}_i \mathbf{x}_i, \end{aligned} \tag{63}$$

- We get measurements related to the state of the obstacles

$$\mathbf{y}_{i,k} = \mathbf{C}_i \mathbf{x}_{i,k} + \mathbf{v}_{i,k}, \tag{64}$$

Problem Formulation

For the trajectory $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ of the ego vehicle ensure

$$\mathcal{P}(\varphi(t)) \cap \left(\bigcup_{i \in \mathcal{I}} \mathcal{O}_i(\varphi_i(t)) \right) = \emptyset, \forall t \in \mathbb{R}_{\geq 0} \tag{65}$$

Safe Navigation Solution

- For this problem, the set-valued flows for the obstacles need to be retrieved from the measurements $\mathbf{y}_{i,k}$.
- Since collision avoidance must be guaranteed, we adopt a set-valued observer.
- Another main challenge is the requirement to have a continuous-time observer such that h is well-defined.
- Our first step is to reduce the ego vehicle to a single point of mass by doing

$$\mathcal{O}_i^+(\mathbf{o}_i) = \mathbf{o}_i + \bar{\mathcal{O}}_i^+ = \mathbf{o}_i + \bar{\mathcal{O}}_i \oplus \left(-\bar{\mathcal{P}}\right), \tag{66}$$

Safe Navigation Solution

- The safety condition becomes

$$\varphi(t) \notin \bigcup_{i \in \mathcal{I}} \mathcal{O}_i^+(\varphi_i(t)) \tag{67}$$

- The obstacle enlarged body evolution in time corresponds

$$\hat{\mathcal{O}}_{i,k}^+(t) = \mathbf{E}_i \left(\mathbf{\Phi}_i(t - t_k)\hat{\mathcal{X}}_{i,k} \oplus \mathbf{\Gamma}_i(t - t_k)\tilde{\mathcal{W}}_i \right) \oplus \bar{\mathcal{O}}_i^+, \tag{68}$$

- Where

$$\mathbf{\Phi}_i(s) = \exp(\mathbf{F}_i s), \quad \mathbf{\Gamma}_i(s) = \frac{\exp(\|\mathbf{F}_i\| s) - 1}{\|\mathbf{F}_i\|} \max_{\mathbf{w}_i \in \mathcal{W}_i} \|\mathbf{w}_i\| \tag{69}$$

Safe Navigation Solution

- Ingredients for the solution:
  - We require a set description that enables all set operations;
  - A method to convert the set description into a valid CBF $h_{i,k}(\mathbf{p}, t))$ for each obstacle;
  - Use the LogSumExp trick to convert to a single CBF

$$h_k(\mathbf{p}, t) = -\frac{1}{\beta_k} \ln \left(\sum_{i \in \mathcal{I}} \exp(-\beta_k h_{i,k}(\mathbf{p}, t)) \right) - \frac{b}{\beta_k}, \qquad (70)$$

  - Apply backstepping to handle the higher relative order dynamics.

Set-valued Observer

- The set-valued estimate for the measurement time k is

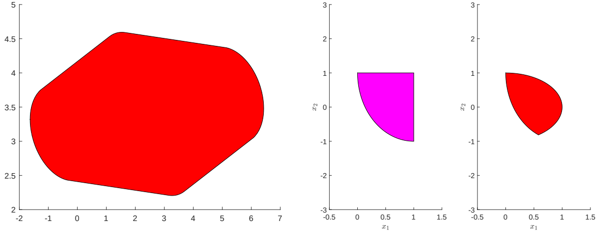
$$\hat{\mathcal{X}}_k = \left(\mathbf{\Phi}(T_s)\hat{\mathcal{X}}_{k-1} \oplus \mathbf{\Gamma}(T_s)\tilde{\mathcal{W}}\right) \cap_{\mathbf{C}} (\mathbf{y}_k - \mathcal{V}), \tag{71}$$

- This means that we require a set description that is exact in:
  - Linear maps, i.e., $\{Rx + t : x \in \mathcal{X}\}$;
  - Minkowski sums, i.e., $\mathcal{X} \oplus \mathcal{Y} = \{x + y : x \in \mathcal{X}, y \in \mathcal{Y}\}$;
  - Generalized intersections, i.e., $\mathcal{X} \cap_R \mathcal{Y} = \{x : x \in \mathcal{X}, Rx \in \mathcal{Y}\}$;
  - Can represent polytopes, ellipsoids and their intersections.

Constrained Convex Generator

A Constrained Convex Generator $\mathcal{CCG}$ is an implicit representation where the generator variables are constrained to some convex sets:

$$\mathcal{CCG} = \{G\xi + c : A\xi = b, \xi \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_{n_p}\}. \tag{72}$$



Nice CCG Properties

- Linear maps, Minkowski sums, and generalized intersections share the same structure as Constrained Zonotopes.

Convex hulls

$$G_h = \begin{bmatrix} G_x & G_y & c_x - c_y \end{bmatrix}, c_h = \frac{c_x + c_y}{2},$$
$$A_h = \begin{bmatrix} A_x & 0 & -b_x \\ 0 & A_y & b_y \end{bmatrix}, b_h = \begin{bmatrix} \frac{1}{2} b_x \\ \frac{1}{2} b_y \end{bmatrix} \quad (73)$$
$$\mathfrak{C}_h = \{\mathfrak{C}_x^{(\tau_x+1)}(\xi_x, \xi_\lambda, -1, 0.5), \mathfrak{C}_y^{(\tau_y+1)}(\xi_y, \xi_\lambda, 1, 0.5), \mathbb{R}\},$$

- Similar result for the Pontryagin difference appearing in ACC 2026.

Convex Set Representations

- an interval corresponds to $(G, c, [\,], [\,], \|\xi\|_\infty \leq 1)$, for a diagonal matrix G;
- a zonotope is given by $(G, c, [\,], [\,], \|\xi\|_\infty \leq 1)$;
- an ellipsoid is defined by $(G, c, [\,], [\,], \|\xi\|_2 \leq 1)$, for a square matrix G;
- a CZ or polytope is $(G, c, A, b, \|\xi\|_\infty \leq 1)$;
- a convex cone in $\mathbb{R}^n$ is $(G, c, [\,], [\,], \xi \geq 0)$;
- ellipsotopes are given by $(G, c, A, b, \|\xi\|_{p_1} \leq 1, \cdots, \|\xi\|_{p_m} \leq 1)$, for some $p_i > 0$, $1 \leq i \leq m$;
- AH-polytopes are given by $(G, c, [\,], [\,], A\xi \leq b)$.

Explicit Finite-Horizon Estimator

- The recursive observer has a number of generators and constraints that is linearly increasing.
- With a fixed horizon, the estimate for in-between samples is given by the CCG

$$
\begin{aligned}
\hat{\mathcal{O}}_k^+(t) &= \left(\mathbf{G}_{\hat{o},k}^+(t), \mathbf{c}_{\hat{o},k}^+(t), \mathbf{A}_{\hat{o},k}^+, \mathbf{b}_{\hat{o},k}^+, \mathfrak{G}_{\hat{o},k}^+\right), \\
\mathbf{G}_{\hat{o},k}^+(t) &= \begin{bmatrix} \mathbf{E}\boldsymbol{\Phi}(t-t_k)\mathbf{G}_{\hat{x},k} & \mathbf{E}\boldsymbol{\Gamma}(t-t_k)\mathbf{G}_{\tilde{w}} & \mathbf{G}_{\bar{o}}^+ \end{bmatrix}, \\
\mathbf{c}_{\hat{o},k}^+(t) &= \mathbf{E}\boldsymbol{\Phi}(t-t_k)\mathbf{c}_{\hat{x},k} + \mathbf{E}\boldsymbol{\Gamma}(t-t_k)\mathbf{c}_{\tilde{w}} + \mathbf{c}_{\bar{o}}^+, \\
\mathbf{A}_{\hat{o},k}^+ &= \begin{bmatrix} \mathbf{A}_{\hat{x},k} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \\
\mathbf{b}_{\hat{o},k}^+ &= \mathbf{b}_{\hat{x},k}, \\
\mathfrak{G}_{\hat{o},k}^+ &= \mathfrak{G}_{\hat{x},k} \times \mathfrak{G}_{\tilde{w}} \times \mathfrak{G}_{\bar{o}}^+,
\end{aligned}
\tag{74}
$$

CBFs for CCGs

- From the fixed-horizon observer, each obstacle is exactly described by a CCG

$$\begin{aligned} \mathcal{O}(t) &= \{\mathbf{G}(t)\boldsymbol{\xi} + \mathbf{c}(t) : \mathbf{A}\boldsymbol{\xi} = \mathbf{b}, \boldsymbol{\xi} \in \mathfrak{G}\} \\ &= (\mathbf{G}(t), \mathbf{c}(t), \mathbf{A}, \mathbf{b}, \mathfrak{G}), \end{aligned} \tag{75}$$

- First, write the parametric solution to the linear equation

$$\boldsymbol{\xi} = \mathbf{A}^{\dagger}\mathbf{b} + \mathbf{N}_{\mathbf{A}}\boldsymbol{\eta}, \tag{76}$$

- The CCG is rewritten as

$$\mathcal{O}(t) = \left(\mathbf{G}(t)\mathbf{N}_{\mathbf{A}}, \mathbf{c}(t) + \mathbf{G}(t)\mathbf{A}^{\dagger}\mathbf{b}, [\,], [\,], \mathfrak{G}'\right). \tag{77}$$

CBFs for CCGs

- The new format of the generator space constraints $\mathfrak{G}'$ is

$$\mathfrak{G}' = \{\boldsymbol{\eta} \in \mathbb{R}^{\eta} : f_j(\boldsymbol{\eta}) \leq 0 \text{ for all } j \in \mathcal{J}\}, \tag{78}$$

- With selector matrices $\mathbf{S}_j$

$$f_j(\boldsymbol{\eta}) = g_j\left(\mathbf{S}_j \mathbf{A}^{\dagger}\mathbf{b} + \mathbf{S}_j \mathbf{N}_{\mathbf{A}} \boldsymbol{\eta}\right) \tag{79}$$

- The set also admits the form $\mathfrak{G}' = \{\boldsymbol{\eta} \in \mathbb{R}^{\eta} : \max_{j \in \mathcal{J}} f_j(\boldsymbol{\eta}) \leq 0\}$, so use the LogSumExp

$$f(\boldsymbol{\eta}) = \frac{1}{\gamma} \ln \left(\sum_{j \in \mathcal{J}} \exp(\gamma f_j(\boldsymbol{\eta})) \right) - \frac{\ln(G+1)}{\gamma}, \tag{80}$$

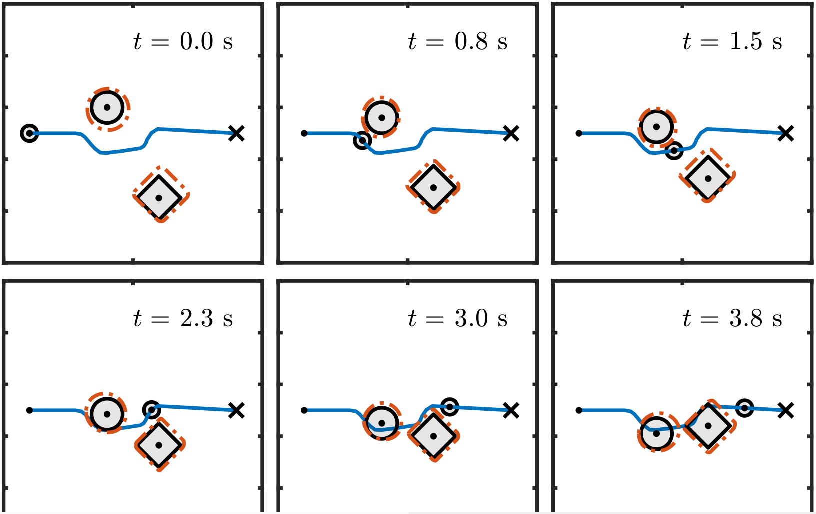
CBFs for CCGs

- Under mild assumptions, a valid CBF representing the CCG is

$$\begin{aligned} h(\mathbf{p}, t) = \min_{\boldsymbol{\eta} \in \mathbb{R}^{\eta}} \; & f(\boldsymbol{\eta}) \\ \text{subject to } & \tilde{\mathbf{G}}(t)\boldsymbol{\eta} + \tilde{\mathbf{c}}(t) = \mathbf{p}, \end{aligned} \tag{81}$$

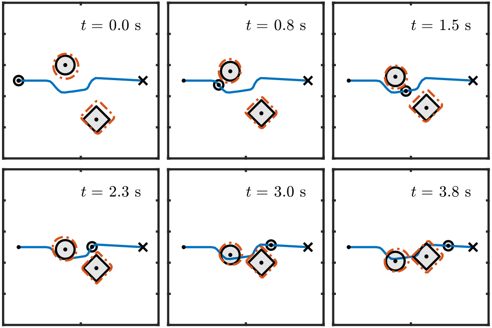
- In practice, this construction was taking an average time of $3\,$ms per obstacle per sampling step using a built-in quasi-Newton in MATLAB.

Simulations on Single-Integrator Ego Vehicle



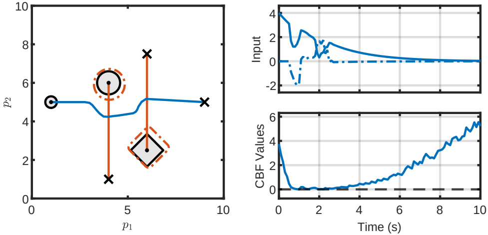
Simulations on Single-Integrator Ego Vehicle

- Consider $\dot{\mathbf{p}} = \mathbf{z}$ with nominal controller $\mathbf{k}_{\mathrm{d}}(\mathbf{p}) = K(\mathbf{p} - \bar{\mathbf{p}})$.
- Ego vehicle has ellipsoidal shape with two obstacles with polytopic and ellipsoidal shapes.



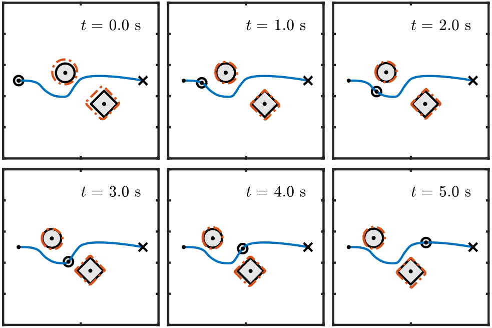
Simulations on Single-Integrator Ego Vehicle

- Ego vehicle always maintains a positive evaluation of the overall CBF.
- Ego vehicle is respecting the uncertainty associated with the set-valued estimation.



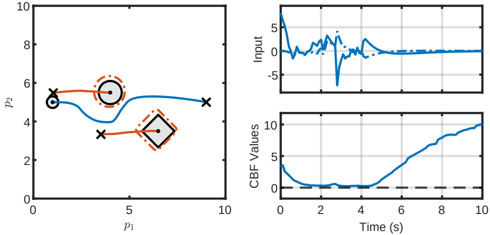
Simulations on Second-order Strict-Feedback Ego Vehicle

- Consider $\dot{\mathbf{p}} = \mathbf{z}, \quad \dot{\mathbf{z}} = (\|\mathbf{p}\| + 0.1)^{-3}\mathbf{p} + \mathbf{u}$, with nominal controller $\mathbf{k}_{\mathrm{d},1}(\mathbf{p}, \mathbf{z}) = -\mathbf{f}_1(\mathbf{p}) + K_1(\mathbf{z} - K(\mathbf{p} - \bar{\mathbf{p}}))$.
- Obstacles are double integrators with random acceleration.



Simulations on Second-order Strict-Feedback Ego Vehicle

- Ego vehicle always maintains a positive evaluation of the overall CBF.
- The observer will produce guaranteed state estimates including all possible acceleration vectors.
- We were using a horizon $N = 5$ with the construction of the CBF taking an average of $8\,$ms per obstacle per sampling step.



Concluding Remarks

- Safe Reorientation and problems with bounded variables have principled ways to assess CBF validity.
- Using a Hybrid CLF-CBF controller can guarantee global asymptotic stability.
- For in-orbit collision avoidance, a guaranteed fixed-horizon observer can be used to produce the safety set-valued flows.
- These set-valued flows can be converted to valid CBFs and applied to systems in strict feedback form.

Student Contributions

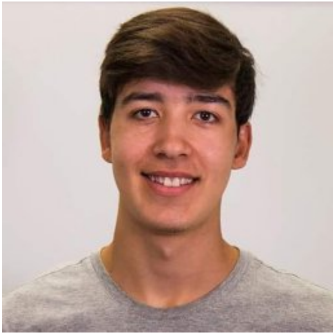


Figure: Hugo Matias (PhD student) - for the topics related to Hybrid CLF-CBF controllers.

Figure: Francisco Rego (former PostDoc) - for the topics related to explicit CCG observers.

References I

[1] H. Matias and D. Silvestre, "Hybrid lyapunov and barrier function-based control with stabilization guarantees," *arXiv preprint arXiv:2504.09760*, 2025.

[2] H. Matias and D. Silvestre, "Safe navigation under uncertain obstacle dynamics using control barrier functions and constrained convex generators," *arXiv preprint arXiv:2601.07715*, 2026.

[3] F. Rego and D. Silvestre, "Explicit computation of guaranteed state estimates using constrained convex generators," in *2024 IEEE 63rd Conference on Decision and Control (CDC)*, 2024, pp. 1400–1405. DOI: 10.1109/CDC56724.2024.10885976

The end

- Thank you for your time.

Safe GNC Design to Handle Polytopic Obstacles, Pointing Constraints, and Other Spacecraft

D. Silvestre

Seminar at EuroGNC
Madrid, Spain

May 8th 2026