CBFs Design with Convex Optimization

Han Wang

EuroGNC CBF Workshop

Acknowledgement

Antonis Papachristodoulou (Oxford)

Kostas Margellos (Oxford)

Claudio De Persis (Groningen)

Paul Goulart (Oxford)

Marta Fochesato (ETH)

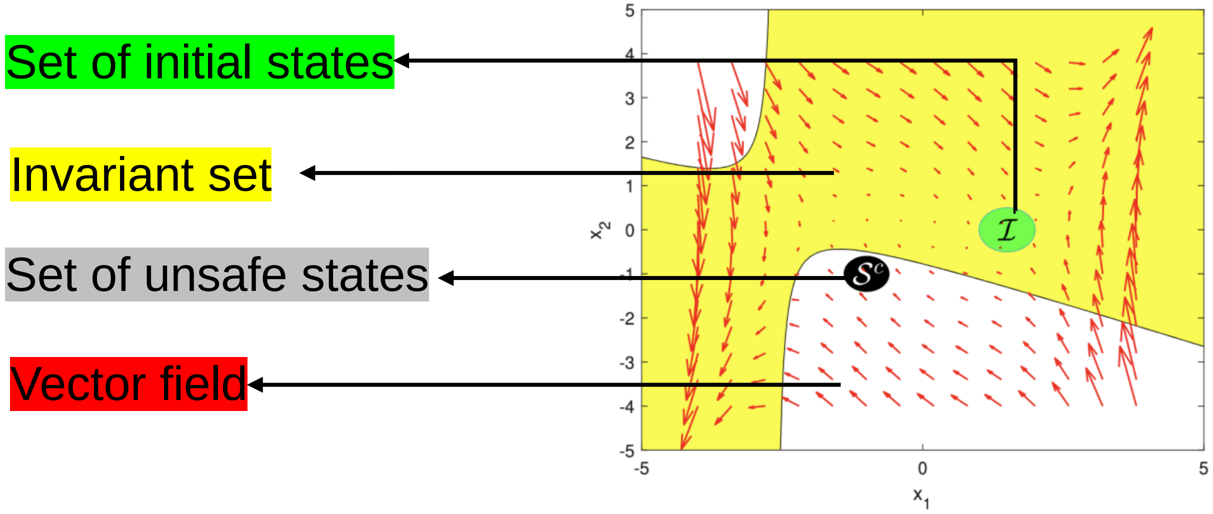
Safety Certificate: Control Barrier Functions

Oxford Control Group
UNIVERSITY OF OXFORD

- Consider system $\dot{x} = f(x, u)$, a set of initial states $\mathcal{I}$, and a set of safe states $\mathcal{S}$

Control Barrier Functions (CBFs)

- Scalar functions defined on state space $b(x) : \mathbb{R}^n \to \mathbb{R}$, differentiable functions
- Set inclusion for the initial conditions $\mathcal{I} \subseteq \mathcal{B} := \{x : b(x) \geq 0\}$
- Set inclusion for the safe states $\mathcal{B} := \{x : b(x) \geq 0\} \subseteq \mathcal{S} := \{x : s(x) \geq 0\}$, <u>$s(x)$ is called the safety function</u>
- Control invariance: $\forall x, \text{such that } b(x) = 0, \exists u : \dot{b}(x) = \frac{\partial b(x)}{\partial x} f(x, u) \geq 0$



Important features:

- On the boundary $\partial\mathcal{B}$, the vector field points inwards $\mathcal{B}$
- The initial states all lie within $\mathcal{B}$
- $\mathcal{B}$ excludes the unsafe sets

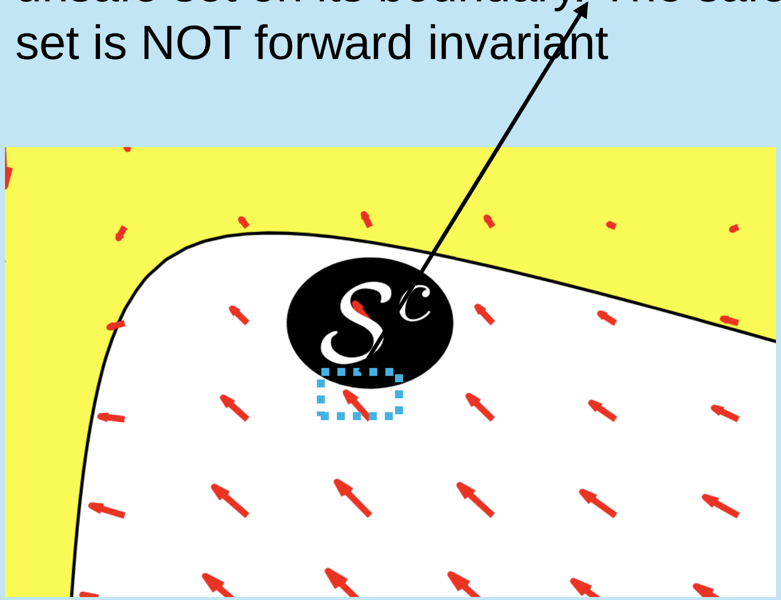
Safety: all trajectories starting from $\mathcal{I}$ will never enter the unsafe set $\mathcal{S}^c$

Are safety Functions Always CBFs?

In set language: are safe sets always forward (for auto. sys.)/control (for ctrl. sys.) invariant?

Autonomous systems: always NO

- Vector field can points inwards the unsafe set on its boundary. The safe set is NOT forward invariant



Control systems: unclear

- Many existing works use the safety functions $s(x)$ as the control barrier functions $b(x)$ for safe control design

$$u^*(x) = \arg\min_{u \in \mathcal{U}} \frac{1}{2}||u||_2^2$$
$$\text{subject to } \underbrace{\frac{\partial s(x)}{\partial x} f(x,u)}_{\dot{s}(x)} + \alpha s(x) \geq 0.$$

- From optimization point of view, the above optimization problem might be infeasible at some state x

Safety Function ≠ CBFs: Pathological Vector Field

Oxford Control Group

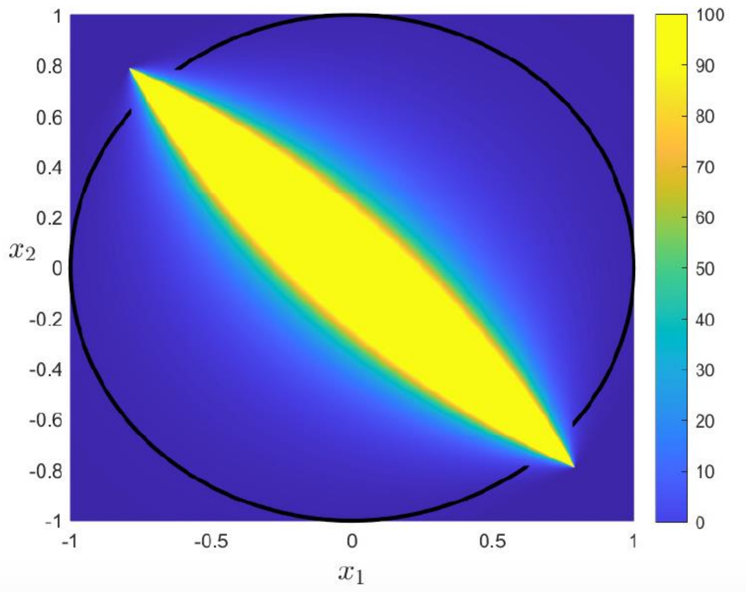
UNIVERSITY OF OXFORD

- Consider a controllable linear system and a safe set

$$\dot{x} = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \quad \mathcal{S} = \{x \in \mathbb{R}^2 : s(x) = x_1^2 + x_2^2 - 1 \geq 0\}$$

If we use $s(x)$ as a “CBF” for controller design…

$$u_s(x) = \min u^\top u, \quad \text{subject to } \dot{s}(x) + 10s(x) \geq 0. \quad \longrightarrow \quad u_s(x) = -\frac{\min\{0, 4x_1^2 - x_1x_2 + 4x_2^2 - 5\}}{x_1 + x_2},$$



The figure show the values of $\|u_s(x)\|^2$ for $-1 \leq x_1 \leq 1, -1 \leq x_2 \leq 1$. The Value is limited to 100 for visualization. The controller is only locally smooth, and the Lipschitz constant is large in a local region as the value varies a lot with little state changes. In other word, the vector field is *pathological*.

Safety Function ≠ CBFs: Mixed Relative-degree

- Consider a linear system and a safe set:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ u_1 \\ u_2 \end{bmatrix}, \quad \mathcal{S} = \{x \in \mathbb{R}^3 : s(x) = x_1^2 + x_2^2 - 1 \geq 0\}$$

The relative degree between $s(x)$ and the system dynamics is mixed:

$$\dot{s}(x) = 2x_1(x_2 + x_3) + 2x_2(x_1 + u_1)$$

Input u_1 appears in $\dot{s}(x)$ for any $x_2 \neq 0$, but u_2 does not appear. When $x_2 = 0, x_1 = 1$ and $x_3 < 0$, we have $s(x) = 0$ but $\dot{s}(x) < 0$.

$$\ddot{s}(x) = 2(2x_2 + x_3)^2 + 4x_1(x_1 + u_1) + 2x_1(x_1 + u_2) + 2(x_1 + u_1)u_1 + 2x_2\dot{u}_1$$

Input u_2 appears in $\ddot{s}(x)$ for any $x_1 \neq 0$, but the terms u_1^2 and $\dot{u}_1$ make it hard to directly apply high-order (exponential) CBF techniques. Backstepping is required!

Design CBFs are necessary for safety!

CBF Design for Linear Systems

Consider linear continuous-time systems of the form

$$\dot{x} = Ax + Bu$$

- $x \in \mathbb{R}^n$ is the state, and $u \in \mathbb{R}^m$ is the control input
- The safe set is specified as $\mathcal{S} \coloneqq \{x \in \mathbb{R}^n : s(x) \geq 0\} \subseteq \mathbb{R}^n$, with $s(x): \mathbb{R}^n \to \mathbb{R}$ be the safety function

We would like to co-design a controller $u(x): \mathbb{R}^n \to \mathbb{R}^m$ and a CBF $b(x): \mathbb{R}^n \to \mathbb{R}$ that guarantee (i) forward invariance of a set $\mathcal{B} \coloneqq \{x \in \mathbb{R}^n : b(x) \geq 0\}$ and (ii) safety $\mathcal{B} \subseteq \mathcal{S}$.

CBF design problem: for a given $s(x)$, design a continuously differentiable function $b(x)$ and a locally Lipschitz continuous controller $u(x)$ such that

$$\mathcal{B} = \{x \in \mathbb{R}^n : b(x) \geq 0\} \subseteq \{x \in \mathbb{R}^n : s(x) \geq 0\} = \mathcal{S} \text{ and}$$

$$\{x \in \mathbb{R}^n : b(x) = 0\} \subseteq \left\{x \in \mathbb{R}^n : \frac{\partial b(x)}{\partial x}(Ax + Bu(x)) \geq 0\right\}$$

Even verifying CBFs for linear systems are generally NP-hard [2]. The goal is to efficiently design CBFs with convex optimisation by *proper parameterizations*.

[1] **H. Wang**, et al., *CDC* 2024, *IEEE TAC* 2025. [2] A. Clark, *IEEE TAC* 2024.

Safe Set $\mathcal{S}$ Specifications

We split the state space $\mathbb{R}^n$ into the direct sum of two subspaces, i.e., $\mathbb{R}^n = \mathbb{R}^{\bar{n}} \oplus \mathbb{R}^{\underline{n}}$,

where $\bar{n} + \underline{n} = n$. Consistently, we partition $x \in \mathbb{R}^n$ as $x = [\bar{x}^\top \; \underline{x}^\top]^\top$.

<u>Safe set $\mathcal{S}$</u> We consider the safe set

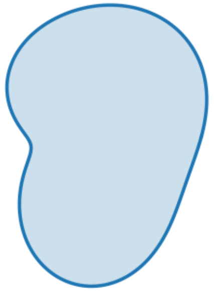
$$\mathcal{S} = \{x \in \mathbb{R}^n : s(x) \geq 0\}$$

and a quadratic safety function

$$s(x) = (\bar{x} - \bar{c})^\top S(\bar{x} - \bar{c}) - 1$$

where $\bar{x} \in \mathbb{R}^{\bar{n}}, 0 \prec S = S^\top \in \mathbb{R}^{\bar{n}\times\bar{n}}$ and $c \in \mathbb{R}^{\bar{n}}$ are known. <u>The unsafe set is bounded on</u> $\mathbb{R}^{\bar{n}}$.

Note: The analysis would only slightly change and lead a more complex convex program if $\mathcal{S}$ is a semi-algebraic set, e.g. set defined by polynomial inequalities/equations.



Control Barrier Functions Parameterization

Oxford Control Group

UNIVERSITY OF OXFORD

CBF design problem: for a given $s(x)$, design a continuously differentiable function $b(x)$ and a locally Lipschitz continuous controller $u(x)$ such that

$$\mathcal{B} = \{x \in \mathbb{R}^n : b(x) \geq 0\} \subseteq \{x \in \mathbb{R}^n : s(x) \geq 0\} = \mathcal{S} \text{ and}$$

$$\{x \in \mathbb{R}^n : b(x) = 0\} \subseteq \left\{x \in \mathbb{R}^n : \frac{\partial b(x)}{\partial x}(Ax + Bu(x)) \geq 0\right\}$$

Motivated by convex CLFs design, we look for a CBF

$$b(x) = (x - c)^\top \Omega^{-1}(x - c) - 1$$

with $\Omega \in \mathbb{R}^{n \times n}$ nonsingular and to be designed. c represents the ”center” of obstacle, which requires a transformation of coordinate for the linear system (similar to CLFs with non-zero equilibria)

Conditions: There exists $\underline{c} \in \mathbb{R}^{\underline{n}}$ be such that $c := [\bar{c} \;\; \underline{c}] \in \mathbb{R}^n$, satisfies

$$\text{rank}([B \;\; Ac]) = \text{rank}(B)$$

We extend $\bar{c}$ into a point $c \in \mathbb{R}^n$ for which there exists an input $d \in \mathbb{R}^m$ such that

$$Ac + Bd = 0$$

Conditions for Set Inclusion $\mathcal{B} \subseteq \mathcal{S}$

Recall that

$$\mathcal{B} = \{x \in \mathbb{R}^n : (x - c)^\top \Omega^{-1} (x - c) - 1 \geq 0\}$$

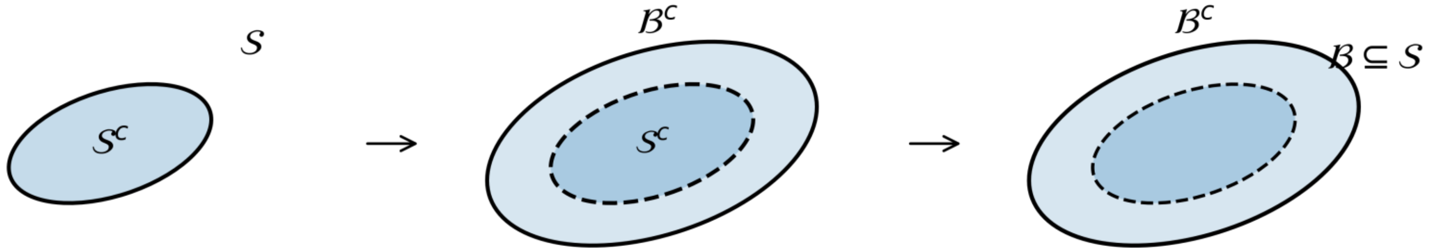
$$\mathcal{S} = \{x \in \mathbb{R}^n : (\overline{x} - \overline{c})^\top S (\overline{x} - \overline{c}) - 1 \geq 0\}$$

where S is given and Ω must be designed

If Ω satisfies

$$\Omega = \begin{bmatrix} \overline{\Omega} & 0_{\overline{n} \times \underline{n}} \\ 0_{\underline{n} \times \overline{n}} & \underline{\Omega} \end{bmatrix}, \qquad \begin{matrix} \overline{\Omega} = \overline{\Omega}^\top \\ -\underline{\Omega} = -\underline{\Omega}^\top \end{matrix} \succ 0, \qquad \begin{bmatrix} S & I_{\overline{n}} \\ I_{\overline{n}} & \overline{\Omega} \end{bmatrix} \succeq 0.$$

then $\mathcal{B} \subseteq \mathcal{S}$



Conditions for Making $\mathcal{B}$ Forward Invariant

Recall the forward invariance condition

$$\{x \in \mathbb{R}^n : b(x) = 0\} \subseteq \left\{x \in \mathbb{R}^n : \frac{\partial b(x)}{\partial x}(Ax + Bu(x)) \geq 0\right\}.$$

We formulate it as $\exists \alpha \in \mathbb{R}_{\geq 0}$ such that $\frac{\partial b(x)}{\partial x}\big(Ax + Bu(x)\big) \geq -\alpha b(x)$ for all $x \in \mathbb{R}^n$. Consider

$$b(x) = (x - c)^\top \Omega^{-1}(x - c) - 1$$

$$u(x) = K(x - c) + d$$

where $Ac + Bd = 0$. This yields the closed-loop system as

$$\dot{x} = Ax + B(K(x - c) + d) = (A + BK)(x - c)$$

The forward invariance condition becomes

$$\begin{bmatrix} x - c \\ 1 \end{bmatrix}^\top \begin{bmatrix} \Omega^{-1}(A + BK) + (A + BK)^\top \Omega^{-1} + \alpha\Omega^{-1} & 0 \\ 0 & -\alpha \end{bmatrix} \begin{bmatrix} x - c \\ 1 \end{bmatrix} \geq 0 \quad \text{for all } x \in \mathbb{R}^n.$$

Conditions for Making $\mathcal{B}$ Forward Invariant

$$\begin{bmatrix} x-c \\ 1 \end{bmatrix}^\top \begin{bmatrix} \Omega^{-1}(A+BK)+(A+BK)^\top\Omega^{-1}+\alpha\Omega^{-1} & 0 \\ 0 & -\alpha \end{bmatrix} \begin{bmatrix} x-c \\ 1 \end{bmatrix} \geq 0 \quad \text{for all } x \in \mathbb{R}^n.$$

This is equivalent to

$$\alpha = 0 \text{ and } \Omega^{-1}(A+BK)+(A+BK)^\top\Omega^{-1} \succcurlyeq 0.$$

Using standard change-of-variables trick to convexify the conditions:

$$\begin{cases} A\Omega + BY + \Omega^\top A^\top + Y^\top B^\top \succeq 0, \\ \alpha = 0. \end{cases}$$

Maximize the size of $\mathcal{B}$ on $\mathbb{R}^{\bar{n}}$: consider the projection of $\mathcal{B}$ on $\mathbb{R}^{\bar{n}}$:

$$\mathcal{PB} = \left\{ \bar{x} \in \mathbb{R}^{\bar{n}} : (\bar{x}-\bar{c})^\top \bar{\Omega}^{-1}(\bar{x}-\bar{c}) - 1 \geq 0 \right\}.$$

Take its complement and minimize its size. As a measure of the size of the ellipsoid, we take

$$trace(\bar{\Omega})$$

A Convex Formulation for CBFs Design

Consider the system $\dot{x} = Ax + Bu$ and the safe set

$$\mathcal{S} := \{x \in \mathbb{R}^n : (\overline{x} - \overline{c})^\top S(\overline{x} - \overline{c}) - 1 \geq 0\}$$

with $S = S^\top \succ 0$ and $\overline{c} \in \mathbb{R}^{\overline{n}}$ given.

Theorem: Suppose there exists $\underline{c} \in \mathbb{R}^{\underline{n}}$ such that $c := \begin{bmatrix} \overline{c} \\ \underline{c} \end{bmatrix}$ satisfies rank($[B \;\; Ac]$) = rank(B). Then compute $d \in \mathbb{R}^m$ such that $Ac + Bd = 0$

$$\begin{aligned} \underset{\Omega, \bar{\Omega}, \underline{\Omega}, Y}{\text{minimize}} \quad & \text{trace}(\bar{\Omega}) \\ \text{subject to} \quad & \Omega = \begin{bmatrix} \bar{\Omega} & 0_{\bar{n}\times \underline{n}} \\ 0_{\underline{n}\times\bar{n}} & \underline{\Omega} \end{bmatrix}, \quad \bar{\Omega} = \bar{\Omega}^\top \succ 0, \quad -\underline{\Omega} = -\underline{\Omega}^\top \succ 0, \quad \begin{bmatrix} S & I_{\bar{n}} \\ I_{\bar{n}} & \bar{\Omega} \end{bmatrix} \succeq 0, \\ & A\Omega + BY + \Omega^\top A^\top + Y^\top B^\top \succeq 0. \end{aligned}$$

then $u(x) = Y\Omega^{-1}(x - c) + d$ makes the set

$$\mathcal{B} = \{x \in \mathbb{R}^n : (x - c)^\top \Omega^{-1}(x - c) - 1 \geq 0\}$$

- forward invariant for the system $\dot{x} = Ax + Bu(x)$ and
- satisfy $\mathcal{B} \subseteq \mathcal{S}$

Simple Example: Pointmass Vehicle on a Line

$$\begin{cases} \dot{\overline{x}} = \underline{x}, \\ \dot{\underline{x}} = u, \end{cases}$$

$\overline{x} \in \mathbb{R}$ represents the position
$\underline{x} \in \mathbb{R}$ represents the velocity

Safe set defined as

$$\mathcal{S} = \{x \in \mathbb{R}^2 : \overline{x}^2 - 1 \geq 0\} \; (S = 1, \overline{c} = 0)$$

CBF and controller parameterized as

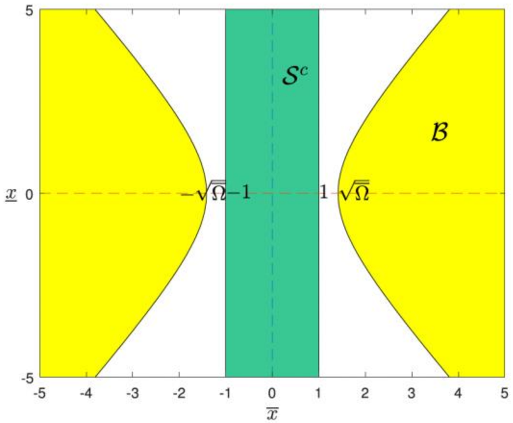
$$b(x) = \frac{\overline{x}^2}{\overline{\Omega}} + \frac{\underline{x}^2}{\underline{\Omega}}, \qquad u(x) = \frac{Y_1 \overline{x}}{\overline{\Omega}} + \frac{Y_2 \underline{x}}{\underline{\Omega}}$$

The forward invariance condition
$\Omega A^\top + A\Omega + Y^\top B^\top + BY \succcurlyeq 0$ is

$$\begin{bmatrix} 0 & \underline{\Omega} + Y_1 \\ \star & 2Y_2 \end{bmatrix} \succeq 0 \iff Y_2 \geq 0, \; \underline{\Omega} + Y_1 = 0.$$

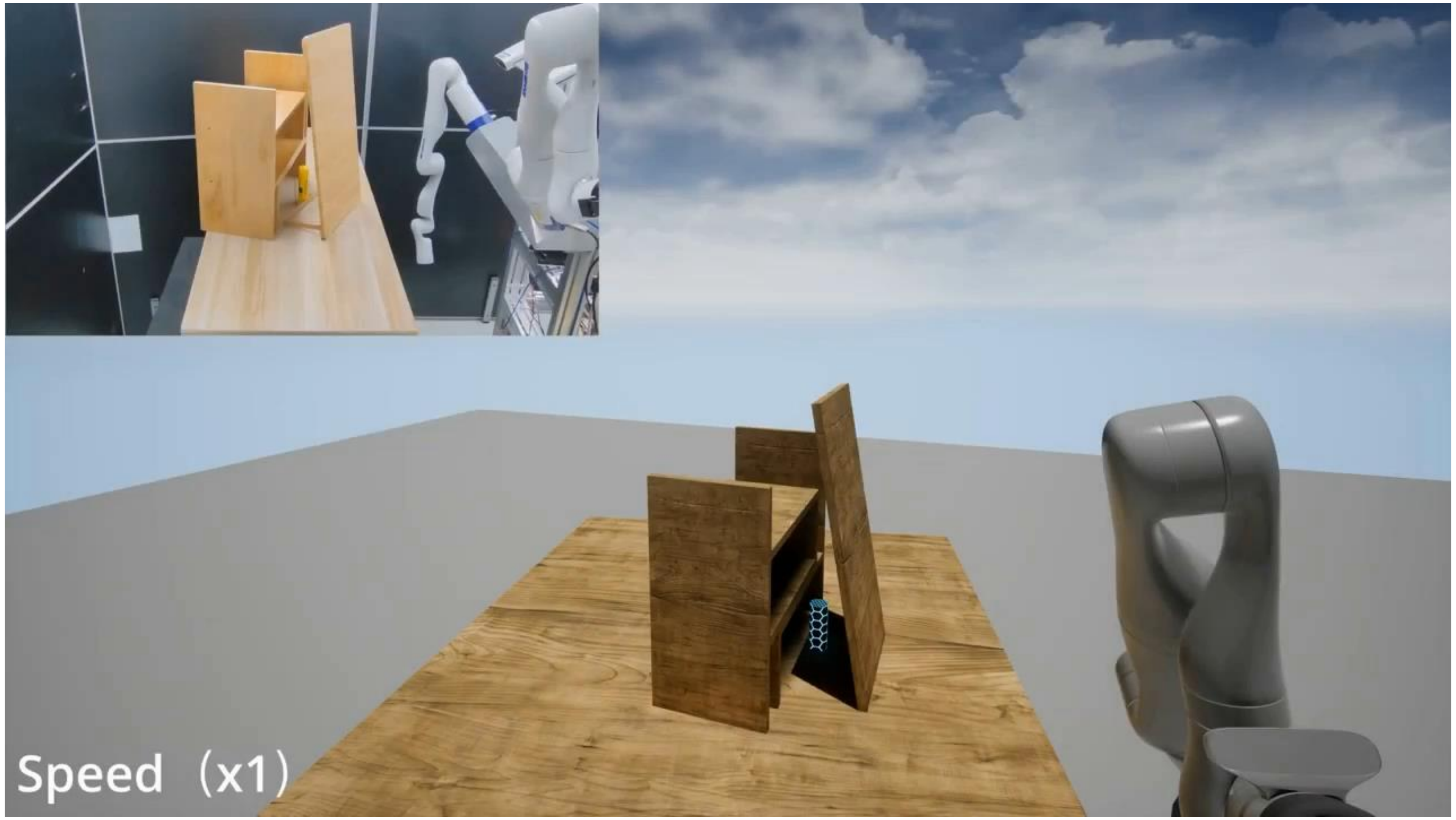
The set inclusion condition $\mathcal{B} \subseteq \mathcal{S}$ is

$$\begin{bmatrix} S & 0_{\overline{n} \times \underline{n}} \\ 0_{\underline{n} \times \overline{n}} & \underline{\Omega} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & \underline{\Omega} \end{bmatrix} \succeq 0.$$



The green set $\mathcal{S}^c \coloneqq \{x \in \mathbb{R}^2 : \overline{x}^2 - 1 \leq 0\}$ is the "unsafe" set. The designed control invariant set $\mathcal{B} \coloneqq \{x \in \mathbb{R}^2 : \overline{\Omega}^{-1} \overline{x}^2 + \underline{\Omega x}^2 - 1 \geq 0\}$ has been filled in yellow.

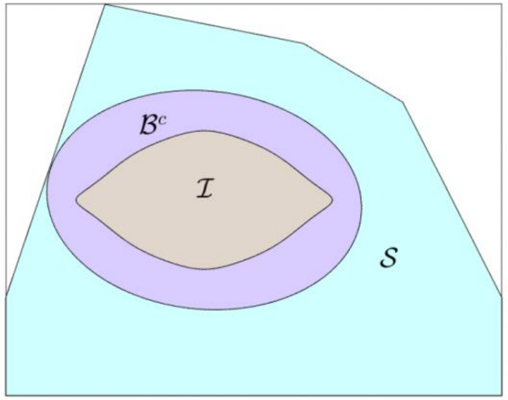
Practical Example: Robotics Manipulator



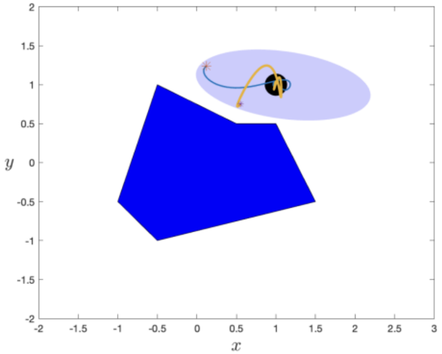
Ref: [1] X. Ding, **H. Wang**, et al., “Online control barrier function construction for safety-critical motion control of manipulators”, *IEEE T-SMC* 2024.

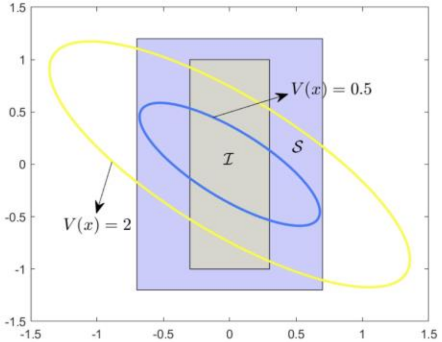
Extensions

- The safe set $\mathcal{S}$ is defined by a convex quadratic function. It can be generalized to semi-algebraic sets, i.e., the sets defined using a finite number of polynomial equations and inequalities, under the assumption that the projection of $\mathcal{S}^c$ onto $\mathbb{R}^{\overline{n}}$ is bounded.
- For the case that $\mathcal{S}$ is bounded, CBF can be still designed as a quadratic function. In this case the trajectories of the closed-loop system will be bounded



I: the initial set. $\mathcal{B}^c$: a bounded control invariant set defined by the CBF. $\mathcal{S}$: the safe set. The trajectories of the closed-loop system will be bounded because of the boundedness of $\mathcal{B}^c$.





- For the case that $\mathcal{S}$ is bounded, bounds on the 2- and ∞- norm as well as polytopic bounds on u can be incorporated into the co-design program as convex constraints.

CBFs Design for Nonlinear Systems

Oxford Control Group UNIVERSITY OF OXFORD

Consider continuous-time nonlinear control affine systems of the form

$$\dot{x} = f(x) + g(x)u$$

- $x \in \mathbb{R}^n$ is the state, and $u \in \mathcal{U} \subseteq \mathbb{R}^m$ is the control input.
- The safe set is specified as $\mathcal{S} := \{x \in \mathbb{R}^n : s(x) \geq 0\} \subseteq \mathbb{R}^n$. The initial set is specified as $I := \{x \in \mathbb{R}^n : w(x) \geq 0\} \subseteq \mathbb{R}^n$, with $s(x), w(x): \mathbb{R}^n \to \mathbb{R}$.

CBF design problem: design $\mathcal{B} = \{x \in \mathbb{R}^n : b(x) \geq 0\}$

1. Initial set is a subset of the control invariant set: $I \subseteq \mathcal{B}$
2. Control invariant set is a subset of the safe set: $\mathcal{B} \subseteq \mathcal{S}$
3. Control invariance: for any x such that $b(x) = 0$, $\exists u \in \mathcal{U}$ such that $\frac{\partial b(x)}{\partial x}(f(x) + g(x)u) \geq 0$

⬇

1. Initial set is a subset of the control invariant set: $b(x) \geq 0, \forall x, s.t. w(x) \geq 0$
2. Control invariant set is a subset of the safe set: $b(x) < 0, \forall x, s.t. s(x) < 0$
3. Control invariance: $\frac{\partial b(x)}{\partial x}(f(x) + g(x)u(x)) \geq 0, \forall x, s.t. b(x) = 0$

Multiplicative coupling

Infinite constraints

Sum-of-squares Relaxation

Problem of interest: positivity of functions over sets.

Sum-of-squares (SOS): positivity over $\mathbb{R}^n$

A multi-variate polynomial $f(x)$ is a sum-of-squares polynomial, termed by *SOS* if there exists m polynomials $f_1(x), \ldots, f_m(x)$ such that

$$f(x) = \sum_{i=1}^{m} f_i(x)^2$$

Lemma: SOS polynomials are nonnegative.

Lemma: $f(x)$ is a SOS polynomial if and only if the following semi-definite program (SDP) is feasible:

$$\begin{aligned} \text{find} \quad & Q \\ \text{s.t.} \quad & f(x) = Z(x)^\top Q Z(x) \\ & Q \succeq 0 \end{aligned} \qquad Z(x) = \begin{bmatrix} x^{\alpha_1} \\ x^{\alpha_2} \\ \vdots \\ x^{\alpha_N} \end{bmatrix}, \qquad x^{\alpha_i} := x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}, \qquad \beta_i \in \mathbb{N}^n.$$

Positivity of Polynomials Over Sets

CBF design problem: design $b(x)$ and $u(x)$ such that
1. Initial set is a subset of the control invariant set: $b(x) \geq 0, \forall x \in \{x \in \mathbb{R}^n : w(x) \geq 0\}$
2. Control invariant set is a subset of the safe set: $b(x) < 0, \forall x \in \{x \in \mathbb{R}^n : s(x) < 0\}$
3. Control invariance: $\frac{\partial b(x)}{\partial x}(f(x) + g(x)u(x)) \geq 0, \forall x \in \{x \in \mathbb{R}^n : b(x) = 0\}$

Assumption: $f(x)$, $g(x)$, $s(x)$ and $w(x)$ are polynomial functions.

Lemma 1: Under the above assumption, let $b(x)$ be a polynomial function. If there exists SOS polynomials $\sigma_1(x), \sigma_2(x) \in SOS[x]$, $\varepsilon > 0$ such that

$$b(x) - \sigma_1(x)w(x) \in SOS[x]$$
$$-b(x) + \sigma_2(x)s(x) - \varepsilon \in SOS[x]$$

Then, it holds that $I := \{x \in \mathbb{R}^n : w(x) \geq 0\} \subseteq \mathcal{B} := \{x \in \mathbb{R}^n : b(x) \geq 0\} \subseteq \mathcal{S} := \{x \in \mathbb{R}^n : s(x) \geq 0\}$

Lemma 2: Under the above assumption, let $b(x)$ and $u(x)$ be polynomial functions. If there exists a polynomial $\lambda_1(x)$, such that

$$\frac{\partial b(x)}{\partial x}(f(x) + g(x)u(x)) + \lambda_1(x)b(x) \in SOS[x]$$

Then, it holds that $\mathcal{B} := \{x \in \mathbb{R}^n : b(x) \geq 0\}$ is forward invariant for $\dot{x} = f(x) + g(x)u(x)$.

SOS Program for CBFs Design

Oxford Control Group
UNIVERSITY OF OXFORD

$$
\begin{aligned}
\text{find} \quad & b(x) \in \mathbb{R}[x]_{d_b}, \quad u(x) \in \mathbb{R}[x]^m_{d_u}, \quad \sigma_1(x), \sigma_2(x) \in SOS[x], \quad \lambda_1(x) \in \mathbb{R}[x]_{d_\lambda} \\
\text{s.t.} \quad & b(x) - \sigma_1(x) w(x) \in SOS[x], \\
& -b(x) + \sigma_2(x) s(x) - \varepsilon \in SOS[x], \\
& \frac{\partial b(x)}{\partial x}\left(f(x) + g(x)u(x)\right) + \lambda_1(x) b(x) \in SOS[x].
\end{aligned}
$$

Multiplicative coupling between $b(x)$ and $u(x)$ makes it nonconvex!

$u^{k+1}(x)$

$$
\begin{aligned}
\text{find} \quad & u^{(k+1)}(x); \lambda_1(x) \\
\text{s.t.} \quad & \frac{\partial b^{(k)}(x)}{\partial x}\left(f(x) + g(x)u^{(k+1)}(x)\right) + \lambda_1(x) b^{(k)}(x) \in SOS[x].
\end{aligned}
$$

$$
\begin{aligned}
\text{find} \quad & b^{(k+1)}(x),\ \sigma_1(x),\ \sigma_2(x),\ \lambda_1(x) \\
\text{s.t.} \quad & \sigma_1(x) \in SOS[x], \qquad \sigma_2(x) \in SOS[x], \\
& b^{(k+1)}(x) - \sigma_1(x) w(x) \in SOS[x], \\
& -b^{(k+1)}(x) + \sigma_2(x) s(x) - \varepsilon \in SOS[x], \\
& \frac{\partial b^{(k+1)}(x)}{\partial x}\left(f(x) + g(x)u^{(k+1)}(x)\right) + \lambda_1(x) b^{(k+1)}(x) \in SOS[x].
\end{aligned}
$$

$b^{k+1}(x)$

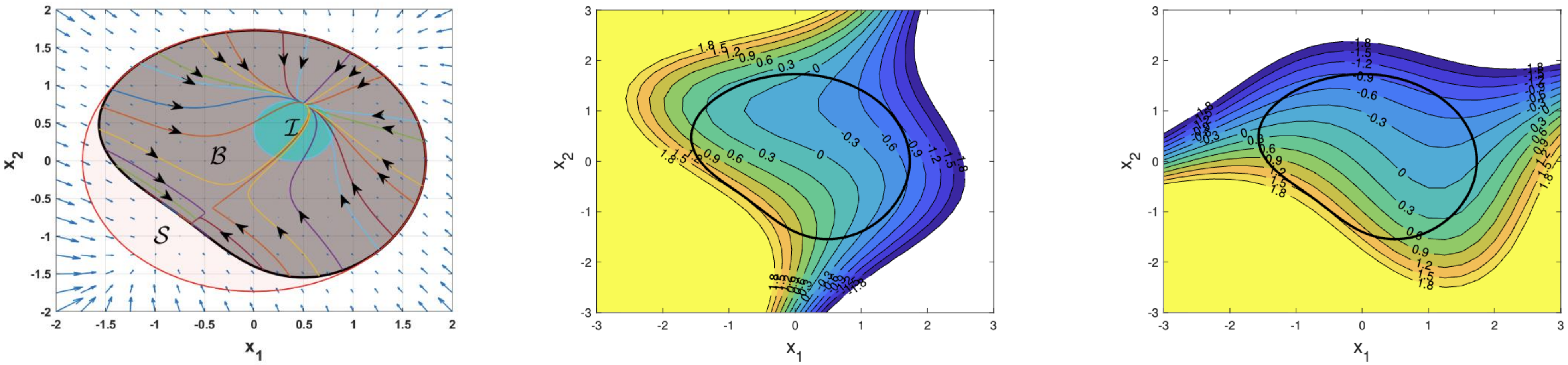
Example: Polynomial System

- Consider the following nonlinear polynomial system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 + \frac{1}{3}x_1^3 + x_2 \end{bmatrix} + \begin{bmatrix} x_1^2 + x_2 + 1 \\ x_2^2 + x_1 + 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

with specification

$$\|u\|_\infty \leq 1.5, \qquad \begin{aligned} \mathcal{I} &= \left\{x \in \mathbb{R}^2 : w(x) := 0.4^2 - \left[(x_1 - 0.4)^2 + (x_2 - 0.4)^2\right] \geq 0\right\}, \\ \mathcal{S} &= \left\{x \in \mathbb{R}^2 : s(x) := x_1^2 + x_2^2 - 1.8^2 \geq 0\right\}. \end{aligned}$$



Extensions & References

Oxford Control Group

UNIVERSITY OF OXFORD

Extensions (with available manuscripts):

1. Discrete-time systems: using rational polynomials.
2. Sampled-based SOS for CBFs design (when the safe set is hard to model).
3. Stochastic design (for bounded and unbounded process noise).
4. Non-polynomial systems: Taylor expansion / recasting into polynomials + equality constraints (for elementary functions, e.g. triangular nonlinear systems).

This presentation:

[1] H. Wang, et al., “Convex Co-Design of Control Barrier Functions and State Feedback Controllers for Linear Systems With Input Constraints”, *CDC* 2024, *TAC* 2025.

[2] H. Wang, et al., “Assessing safety for control systems using sum-of-squares programming”, *Polynomial Optimization, Moments and Applications* 2023 (tutorial paper).

[3] H. Wang, et al., “Safety verification and controller synthesis for systems with input constraints”, *IFAC* 2023.

Conclusions & Future Directions

Conclusions:

- We propose convex optimization (SOS, SDP) based methods for CBFs design.
- Convex formulation is provided for linear systems. Solving one single SDP to co-design a quadratic (but nonconvex) CBF and an affine controller.
- The method is highly efficient and *guaranteed*, in contrast to most learning-based methods that do not incorporate formal verification, e.g. NN verification.

Future directions:

- Exploring sparsity of the SOS program for acceleration.
- Data-driven design for systems without models, or accurate models.
- Time-varying CBFs design.
- …

Thank you!

hanwang1@ethz.ch